Breaking the Dirac code

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Abstract. In describing the fermionic state, the Dirac equation is the most fundamental in physics. The full power of the equation, however, seems to be obscured by a complicated mathematical apparatus which prevents us gaining a more direct knowledge of the fermionic state. To get beyond this hurdle means breaking the equation's mathematical code. The reward for doing this is a new set of insights into particle physics and quantum mechanics.

1 The Dirac code

The Dirac equation is the most fundamental in physics, the only equation which applies universally to the most fundamental particles or fermions. Somehow or other it must contain most of the information which makes up particle physics. But the Dirac equation is a cryptic one – 'a riddle, wrapped in a mystery, inside an enigma' – with a seemingly bizarre mathematical structure. The question is: can we crack the code and find the physics that the structure of the equation obscures? By 'cracking the code', we mean finding an expression of relativistic quantum mechanics which is transparent – which actually provides purely physical information. We will know when we have done it when the mathematics is no longer arbitrary but an intrinsic part of the physical structure.

It is not difficult to see where the problems begin. The equation looks concise but a large part of it (actually 98.4 %) is redundant. It is also scrambled and less symmetric than it at first appears. In

$$(\gamma^{\mu}\partial_{\mu} + im)\ \psi = 0, \qquad (1)$$

or

$$\left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} + im\right) \psi = 0\ , \qquad (2)$$

$\gamma^{\mu}\partial_{\mu}$ is a 4 × 4 matrix, ψ is a 4-component spinor, while each of the γ^{μ} terms is a 4 × 4 matrix. It is certainly the matrices that are the initial problem. So, it is important to establish what exactly the γ^{μ} terms actually do. There are, in fact, five of them, but only four appear in the equation. Dirac introduced them to make quantum mechanics linear in space and time and needed algebraic square roots to do this. He needed a system of five operators in which

$$(\gamma^0)^2 = (\gamma^5)^2 = 1 \qquad\qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$

and *all* terms anticommute with each other: $\gamma^0\gamma^1 = -\gamma^1\gamma^0$, etc.

Now, although Dirac used matrices, there is no compulsion on us to do likewise, as non-matrix anticommuting square roots of –1 and 1 have been around for a century and a half in the respective forms of quaternions and complexified quaternions. The respective multiplication rules for the units of these algebras are:

Quaternions	**Complexified quaternion**
$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = 1$	$(i\boldsymbol{i})^2 = (i\boldsymbol{j})^2 = (i\boldsymbol{k})^2 = i(i\boldsymbol{i})(i\boldsymbol{j})(i\boldsymbol{k}) = 1$
$\boldsymbol{ij} = \square\boldsymbol{ji} = \boldsymbol{k}$	$(i\boldsymbol{i})(i\boldsymbol{j}) = \square(i\boldsymbol{j})(i\boldsymbol{i}) = i(i\boldsymbol{k})$
$\boldsymbol{jk} = \square\boldsymbol{kj} = \boldsymbol{i}$	$(i\boldsymbol{j})(i\boldsymbol{k}) = \square(i\boldsymbol{k})(i\boldsymbol{j}) = i(i\boldsymbol{i})$
$\boldsymbol{ki} = \square\boldsymbol{ik} = \boldsymbol{j}$	$(i\boldsymbol{k})(i\boldsymbol{i}) = \square(i\boldsymbol{i})(i\boldsymbol{k}) = i(i\boldsymbol{j})$.

There are also two significant algebras which are isomorphic to complexified quaternions, with multiplication rules:

Pauli matrices	**Multivariate vectors**
$\sigma_x{}^2 = \sigma_y{}^2 = \sigma_z{}^2 = I$	$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -i\mathbf{ijk} = 1$
$\sigma_x\sigma_y = -\,\sigma_y\sigma_x = i\sigma_x$	$\mathbf{ij} = -\mathbf{ji} = i\mathbf{k}$
$\sigma_y\sigma_z = -\,\sigma_z\sigma_y = i\sigma_x$	$\mathbf{jk} = -\mathbf{kj} = i\mathbf{i}$
$\sigma_z\sigma_x = -\,\sigma_x\sigma_z = i\sigma_y$	$\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}$.

Here, Pauli matrices are defined as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with unit} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

while multivariate vectors are ones with a full product, combining the scalar and vector products:

$$\mathbf{ab} = \mathbf{a.b} + i\,\mathbf{a} \times \mathbf{b}$$

So $\qquad \mathbf{aa} = a^2 \qquad \mathbf{ii} = 1 \qquad \mathbf{ij} = i\mathbf{k}, \qquad$ etc.

An algebra *combining* anticommuting square roots of 1 and –1 has also been around for more than a century, in the form of Clifford or geometrical algebra, symbolized by $Cl(m, n)$ of $G(m, n)$, where m and n are the respective numbers of units with norm 1 and –1.

Pauli matrices are, of course, the basis for the Dirac matrices, but they create intrinsic problems. There are only 2 degrees of freedom, defined by the complex plane. One of the three matrices is defined only as the product of the other two, and so has complex coefficients where the others have real ones. The representation thus becomes asymmetric. Another problem is the skew-symmetry between σ_X and σ_Z. The γ matrices, which are constructed from the Pauli matrices, inherit these problems, as we now have:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inserting these directly into (1), imports the same difficulties into the Dirac equation:

$$\begin{pmatrix} \frac{\partial}{\partial t} + im & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial t} + im & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial t} + im & 0 \\ -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial t} + im \end{pmatrix} \psi = 0$$

This is evident from the free particle amplitudes:

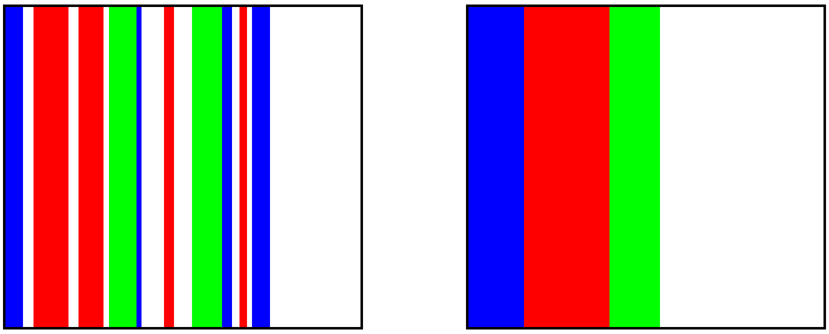
$$\left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad \left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$\left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

The solutions have different phase factors, the first two being positive energy and the second two positive energy, and E and m are inextricably and unphysically mixed. Also, we have terms like $p_X + ip_Y$ and $p_X - ip_Y$, again with no counterpart in nature. In the process, also, we change the momentum-energy or space-time signature from the true + + + – to the distorted + + – –, that is we have two spacelike and two timelike components, meaning that the expressions are not truly relativistic. Among other things, as we will show, this makes it impossible to produce a satisfactory mathematical model for baryonic structure because the $SU(3)$ strong interaction derives from the 3-D symmetry of the momentum operator which this distorted model denies.

2 Defragmenting Dirac

The matrices have proved a problem because they cause fragmentation of the equation, mixing up energy, momentum and mass terms. They also take up too much logical space, requiring 16 pieces of information for one operation. In addition, they lack symmetry. There are 5 terms in the equation, but only 4 have a γ matrix. Yet there is a fifth matrix (γ^5) in the algebra. The first thing we must do is *defragment* – that is, separate energy, momentum and mass terms from each other. We need separate 'bins' for **energy**, **momentum** and **mass**.



In fact, we have created a problem that need not exist. There is no necessity to interpret the γ operators as matrices at all. We can do the same job using Clifford algebra, and this will also solve the logical space problem by reducing the 16 operators of the 4 × 4 matrix to one.

The problem is which? There are many Clifford algebras and many ways of applying it. The only point in applying a new algebra is to crack the code, and unlock the physical information in the equation. But it seems that we need the exact algebra to do this – experience shows that only one will work. Elegant reformulations of Dirac using quaternions and Clifford algebra already exist – but they don't solve the fundamental problem. However, the correct algebra alone will not be enough. There will be no completely deductive mathematical path to unravelling the code, just as there was no deductive path for Dirac in producing his equation.

The algebra we will use is a combination (tensor product) of quaternions and multivariate 4-vectors (equivalent to complex double quaternions). The complete set of units is:

$\boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}$	quaternion units	$\mathbf{i}\ \mathbf{j}\ \mathbf{k}$	multivariate vector units
1	scalar	i	pseudoscalar

This is intriguingly close to twistor algebra, a complex 4-D space-time, now used in QCD, as there are 4 real parts (norm 1) and 4 imaginary (norm –1), though here there are additional specifications.

The two 3-D algebras (quaternion and multivariate vector) are commutative in this algebra. So, there are 64 possible products of the 8 units

$(\pm 1, \pm i)$	4	units
$(\pm 1, \pm i) \times (\mathbf{i}, \mathbf{j}, \mathbf{k})$	12	units
$(\pm 1, \pm i) \times (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$	12	units
$(\pm 1, \pm i) \times (\mathbf{i}, \mathbf{j}, \mathbf{k}) \times (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$	36	units

This is the same as with the γ matrices. However, in this case, a considerable simplification is possible, by effective elimination of the vectors.

Assuming that ± signs are determined by the order of multiplication, the 2 × 32 units can be generated binomially from five composite ones, like the γ matrices. There are numerous ways of doing this, but all basically require distributing the units of one 3-D structure among the rest, e.g.,

i	$\mathbf{i}\ \mathbf{j}\ \mathbf{k}$	1	$\boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}$
$i\boldsymbol{k}$	$\mathbf{i}\boldsymbol{i}\ \mathbf{j}\boldsymbol{i}\ \mathbf{k}\boldsymbol{i}$	$1\boldsymbol{j}$	

Such *pentads* can be easily mapped onto the γ matrices. Two such mappings will be of particular use to us:

$\gamma^0 = -i\boldsymbol{i}$		$\gamma^0 = i\boldsymbol{k}$	
$\gamma^1 = \mathbf{i}\boldsymbol{k}$		$\gamma^1 = \mathbf{i}\boldsymbol{i}$	
$\gamma^2 = \mathbf{j}\boldsymbol{k}$		$\gamma^2 = \mathbf{j}\boldsymbol{i}$	
$\gamma^3 = \mathbf{k}\boldsymbol{k}$		$\gamma^3 = \mathbf{k}\boldsymbol{i}$	
$\gamma^5 = i\boldsymbol{j}$	(A)	$\gamma^5 = i\boldsymbol{j}$	(B)

Choosing mapping (A), and substituting into (2), we obtain:

$$\left(-i\boldsymbol{i}\frac{\partial}{\partial t} + \boldsymbol{k}\mathbf{i}\frac{\partial}{\partial x} + \boldsymbol{k}\mathbf{j}\frac{\partial}{\partial y} + \boldsymbol{k}\mathbf{k}\frac{\partial}{\partial z} + im\right)\psi = 0.$$

A key move now is to multiply the equation from the left by $\boldsymbol{j}$. This apparently trivial operation has profound consequences, because we now alter the mapping from (A) to (B), and obtain:

$$\left(i\boldsymbol{k}\frac{\partial}{\partial t} + \boldsymbol{i}\mathbf{i}\frac{\partial}{\partial x} + \boldsymbol{i}\mathbf{j}\frac{\partial}{\partial y} + \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial z} + i\boldsymbol{j}m\right)\psi = 0.$$

The key result is that the equation is now fully *symmetrical*, and the quaternion operators provide the 3 ‘bins’ we require:

$\boldsymbol{k}$	$\boldsymbol{i}$	$\boldsymbol{j}$
energy	momentum	mass

The real significance of the extra symmetry becomes apparent when we try inserting a plane wave solution for ψ:

$$\psi = A\, e^{-i(Et - \mathbf{p.r})}.$$

Then

$$(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{i}p_x + i\boldsymbol{i}\mathbf{j}p_y + i\boldsymbol{i}\mathbf{k}p_z + i\boldsymbol{j}m)\, A\, e^{-i(Et - \mathbf{p.r})} = 0$$

or

$$(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\, A\, e^{-i(Et - \mathbf{p.r})} = 0, \qquad (3)$$

where $\mathbf{p}$ is multivariate. However, (3) is only valid if A is a multiple of ($\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m$). A is a *nilpotent* or square root of zero. This must also be true of ψ, and must have been true *even before we made the substitution of algebraic operators for matrices*.

3 The Dirac 4-spinor

Of course, from the conventional Dirac equation we know that there are really *four* solutions within the 4-component spinor ψ. But with the new symmetrical form of the equation, and with the knowledge (from Hestenes[1]) that a multivariate $\mathbf{p}$ or ∇ already incorporates fermionic spin, these are easy to identify

fermion / antifermion	$\pm E$
spin up / down	$\pm \mathbf{p}$

These options apply to both the amplitude and the phase factor. Here, we will use the convention of describing the term equivalent to A in $A\, e^{-i(Et - \text{p.r})}$ as the 'amplitude' (rather than the entire term) and the term equivalent to $e^{-i(Et - \text{p.r})}$, conventionally known as the 'phase factor', as the 'phase'.

We note also that, for a multivariate $\mathbf{p}$,

$$\mathbf{pp} = (\boldsymbol{\sigma}.\mathbf{p})\,(\boldsymbol{\sigma}.\mathbf{p}) = pp = p^2.$$

So we can also use $\boldsymbol{\sigma}.\mathbf{p}$ for $\mathbf{p}$ (or $\boldsymbol{\sigma}.\nabla$ for ∇) in the Dirac equation, where $\boldsymbol{\sigma}.\mathbf{p}$ is helicity, and $\boldsymbol{\sigma}$ is a pseudovector of magnitude –1. We will only need to explicitly incorporate spin where the vectors are not multivariate (e.g. using polar coordinates). So we can represent the Dirac 4-spinor by a column vector with these 4 components:

$$\psi_1 = (\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\, e^{-i(Et - \text{p.r})}$$
$$\psi_2 = (\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\, e^{-i(Et + \text{p.r})}$$
$$\psi_3 = (-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\, e^{i(Et - \text{p.r})}$$
$$\psi_4 = (-\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\, e^{i(Et + \text{p.r})}$$

each of which is operated on by

$$\left(i\boldsymbol{k}\frac{\partial}{\partial t} + \boldsymbol{i}\nabla + i\boldsymbol{j}m \right).$$

However, by removing the 4 × 4 matrices, we have effectively reduced the differential operator to a single term. This gives us the logical space to add an extra twist of symmetry to the equation, by making the operator a 4-component spinor, identical in representation to the amplitude. To do this, we transfer the variation in the signs of *E* and **p** from the phase term to the differential operator.

$$\left(i\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{t}} + \boldsymbol{i}\nabla + i\boldsymbol{j}m\right)(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0$$
$$\left(i\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{t}} - \boldsymbol{i}\nabla + i\boldsymbol{j}m\right)(\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0$$
$$\left(-i\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{t}} + \boldsymbol{i}\nabla + i\boldsymbol{j}m\right)(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0$$
$$\left(-i\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{t}} - \boldsymbol{i}\nabla + i\boldsymbol{j}m\right)(-\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0$$

We see here that the Feynman representation of negative energy states being associated with negative time is automatically applied. Also, with only a single phase, the expression

$$(\pm\, \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$$

or, more conveniently,

$$(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m),$$

if we adopt a more physically valid sign convention, can be taken to refer either to the differential term in operator form, or to the amplitude produced by applying this to the phase. In any case, the term $(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ remains a nilpotent, and the expression

$$(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0$$

(in which the first term is conveniently a row vector and the second a column vector) becomes a perfect expression of the Pauli exclusion principle.

Pauli exclusion, however, is not unique to free fermions, and this brings us to a step which, in retrospect, might appear revolutionary, and take us into new territory beyond the safety of the conventional Dirac equation. We assume that *all fermionic amplitudes in all states* are nilpotent. We postulate that the most general form for a wavefunction is nilpotent, and that we should seek specifically nilpotent solutions for all problems. We justify this on the grounds that all fermionic states are Pauli exclusive, and that calculations such as that for the hydrogen atom effectively assume this to be the case, but the ultimate justification is that it works. What it means is that the $(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ operator must always produce nilpotent solutions even when *E* and **p** are covariant derivatives or contain field terms.

4 The nilpotent operator

The principle we have adopted is revolutionary because it means thatwe no longer need an equation of any kind. Relativistic quantum mechanics is completely specified by an operator of the form

$$(\pm\, i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)$$

which completely determines the phase factor, or, in our terminology, the 'phase', that will provide a nilpotent amplitude. Finding the solution generally means finding the phase. In fact, we only need the first term of the operator, say $(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)$, as the other three terms are produced by automatic sign changes, and do not represent independent information; and we will now use the convention that this really represents $(\pm\, i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)$. It is difficult to see how fundamental physical information could be more compactified. We have now reduced the information used in the Dirac equation to 1 / 64 of its original amount, and hence shown that 98.4 % is really redundant.

We can see that each complete fermion operator incorporates four individual creation (or annihilation) operators

$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)$	fermion spin up
$(i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$	fermion spin down
$(-i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)$	antifermion spin down
$(-i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$	antifermion spin up

The nature of the state is determined by which of these is the lead term. The others can be regarded as vacuum states representing ones into which it could transform. Because of the way they are defined, nilpotent operators are specified with respect to the entire quantum field. Formal second quantization is unnecessary. We can consider the nilpotency as defining the interaction between the localized fermionic state and the unlocalized vacuum, with which it is uniquely self-dual. The phase is the mechanism through which this is accomplished. Defining a fermion implies simultaneous definition of vacuum as 'the rest of the universe' with which it interacts. The nilpotent structure then provides energy-momentum conservation without requiring the system to be closed. The nilpotent structure is thus naturally *thermodynamic*.

We can now see that the expression

$$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\,(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m) \to 0$$

has at least *five* independent meanings.

classical	special relativity
operator × operator	Klein-Gordon equation
operator × wavefunction	Dirac equation
wavefunction × wavefunction	Pauli exclusion

fermion × vacuum thermodynamics

We now have an operator (***ik**E* + ***i*p** + ***j**m*) that potentially incorporates all the physical information available to the fundamental physical state. If it is no longer 'coded', then this physical information should be transparent. It is easy to show that we can use our operator to do conventional quantum mechanics, e.g. by defining a probability density by multiplying by its complex quaternion conjugate (***ik**E* – ***i*p** – ***j**m*). We can also derive spin ½ in the usual way, and the one-handed helicity of massless fermionic states. And we can proceed to do *zitterbewegung*, QED, QFD, QCD, renormalization, and other calculations.[2,3] However, more profoundly, we can show that the nilpotent structure has implications for the origin of particle structures and the fundamental interactions.

5 Particle states and interactions

If the fermionic nilpotent is the most fundamental structure in physics – in effect, its fundamental unit, can it reproduce the fundamental particle states and their interactions? These two questions are not independent of each other. The first thing is to see if the *structure* of the nilpotent operator can give us any insight into the nature of fermionic interactions. In fact, this is precisely what it can do. But, first, assuming that the constraint of spherical symmetry exists for a point particle, we can express the momentum term of the operator in polar coordinates, using the Dirac prescription, with an explicit spin term:

$$\boldsymbol{\sigma}.\nabla = \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \pm i \, \frac{j + ½}{r} \, .$$

We need the spin term because the multivariate nature of the **p** term cannot be expressed in polar coordinates.

The nilpotent Dirac operator now becomes:

$$\left(\boldsymbol{k}E + \boldsymbol{i}\left(\frac{\partial}{\partial \boldsymbol{r}} + \frac{1}{r} \pm i \, \frac{j + ½}{\mathrm{r}} \right) + \boldsymbol{ij}m \right) .$$

Now, whatever phase we apply this to, we will find that we will not get a nilpotent solution unless the 1 / *r* term with coefficient ***i*** is matched by a similar 1 / *r* term with coefficient ***k***. So, simply requiring *spherical symmetry* for a point particle, requires a term of the form *A* / *r* to be added to *E*. If all point particles are spherically symmetric sources, then the minimum nilpotent operator is of the form

$$\left(\boldsymbol{k}\left(E - \frac{A}{r} \right) + \boldsymbol{i}\left(\frac{\partial}{\partial \boldsymbol{r}} + \frac{1}{r} \pm i \, \frac{j + ½}{\mathrm{r}} \right) + \boldsymbol{ij}m \right) .$$

To establish that this is a nilpotent, we must now find the phase to which this must apply to create a nilpotent amplitude. This will quite quickly produce the characteristic solution for the Coulomb force (i.e. the ‘hydrogen atom’ solution). The solution is straightforward. We apply this to the phase

$$F = e^{-ar} r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu} \;\; ,$$

which we already know, to find the amplitude, and equate the squared amplitude to zero. Here, we obtain:

$$4\left(E - \frac{A}{r}\right)^2 = -2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + \ldots \frac{1}{r} + i\,\frac{j + ½}{\mathrm{r}}\right)^2 - 2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + \ldots \frac{1}{r} - i\,\frac{j + ½}{\mathrm{r}}\right)^2 + 4m^2 \,.$$

Equating constant terms leads to

$$E^2 = -a^2 + m^2 \;\; ,$$

$$a = \sqrt{m^2 - E^2} \;\; .$$

Equating terms in $1/r^2$, with $\nu = 0$, we obtain:

$$\left(\frac{A}{r}\right)^2 = -\left(\frac{\gamma + 1}{r}\right)^2 + \left(\frac{j + ½}{r}\right)^2 ,$$

from which, excluding the negative root (as usual),

$$\gamma = -1 + \sqrt{(j + ½)^2 - A^2} \;\; .$$

Assuming the power series terminates at n', and equating coefficients of $1/r$ for $\nu = n'$,

$$2EA = -2\sqrt{m^2 - E^2}\,(\gamma + 1 + n'),$$

the terms in (j + ½) cancelling over the summation of the four multiplications, with two positive and two negative. From this we may derive

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \dfrac{A^2}{(\gamma + 1 + n')^2}}} = \frac{1}{\sqrt{1 + \dfrac{A^2}{\left(\sqrt{(j + ½)^2 - A^2} + n'\right)^2}}} \;\; .$$

With $A = Ze^2$, we obtain the hyperfine or fine structure formula for a one-electron nuclear atom or ion:

$$\frac{E}{m}=\frac{1}{\sqrt{1+\frac{\left(Ze^2\right)^2}{\left(\gamma+1+n'\right)^2}}}=\frac{1}{\sqrt{1+\frac{\left(Ze^2\right)^2}{\left(\sqrt{\left(j+½\right)^2-\left(Ze^2\right)^2}+n'\right)^2}}}\,.$$

The significance of this result is that we have, without mentioning anything about potentials or interactions, or anything physical at all, and only using the structure of the nilpotent operator, needed to maintain the spherical symmetry of a point-particle source, created the solution for the Coulomb or inverse linear potential. And we have shown that it is absolutely necessary to any fermionic state described as a point source, regardless of what other potentials may be present.

This solution does not require the usual two sets of interacting solutions for positive and negative, as the whole thing arrives as a package in the nilpotent formalism. It results purely from spherical symmetry. Only *two* other nilpotent solutions are possible assuming spherical symmetry. With an additional linear potential ($\propto r$), we obtain the characteristic strong interaction infrared slavery / asymptotic freedom with any other additional potential (e.g. dipole / multipole), we obtain a harmonic oscillator, which we can take to be a weak signature. We can now proceed to show that these two other potentials can, like the Coulomb potential, be derived from the structure of the nilpotent operator alone.

6 Baryons and the strong interaction

A significant test of the validity of the nilpotent structure is the case of the baryon – a set of three interacting states which has no satisfactory representation within the conventional formalism. Conventionally, we consider a baryon to be made up of three fermionic components, to which we assign colour to overcome Pauli exclusion. Can we relate this concept of colour to the fundamental structure of nilpotents?

Clearly, we cannot have a 3-component state vector of the form:

$$(i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)\ (i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)\ (i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)$$

as this zeros automatically. However,

$$\begin{aligned}(i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)\ (i\boldsymbol{k}E + \boldsymbol{j}m)\ (i\boldsymbol{k}E + \boldsymbol{j}m) &\rightarrow (i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)\\ (i\boldsymbol{k}E + \boldsymbol{j}m)\ (i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)\ (i\boldsymbol{k}E + \boldsymbol{j}m) &\rightarrow (i\boldsymbol{k}E - i\mathbf{p} + \boldsymbol{j}m)\\ (i\boldsymbol{k}E + \boldsymbol{j}m)\ (i\boldsymbol{k}E + \boldsymbol{j}m)\ (i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m) &\rightarrow (i\boldsymbol{k}E + i\mathbf{p} + \boldsymbol{j}m)\end{aligned}$$

So it would be possible to have a nonzero state vector if we use the vector properties of **p** and the arbitrary nature of its sign (+ or –).

A state vector of the form, privileging the **p** components:

$$(i\boldsymbol{k}E \pm i\mathbf{i}p_X + \boldsymbol{j}m)\ (i\boldsymbol{k}E \pm i\mathbf{j}p_Y + \boldsymbol{j}m)\ (i\boldsymbol{k}E \pm i\mathbf{k}p_Z + \boldsymbol{j}m)$$

has six independent allowed phases, i.e. when

$$\mathbf{p} = \pm\, \mathbf{i}p_X \,,\; \mathbf{p} = \pm\, \mathbf{j}p_Y \,,\; \mathbf{p} = \pm\, \mathbf{k}p_Z \,,$$

but these must be *gauge invariant*, i.e. indistinguishable, or all present at once. Also, we must interpret the E, **p**, m symbols as belonging to a totally entangled state, rather than the subcomponents. In principle, we could then see the 'quark' structure as using the concept of spatial (rather than temporal) separation to represent the arbitrary nature of the direction of fermionic spin.

One method of picturing the arbitrary nature of the phases (gauge invariance) is to imagine an automatic mechanism of transfer between them.

$$\begin{array}{ll}
(i\mathbf{k}E + \boldsymbol{i}\, \mathbf{i}p_X + \boldsymbol{j}\, m)\, (i\mathbf{k}E + \boldsymbol{i}\, \mathbf{j}p_Y + \boldsymbol{j}\, m)\, (i\mathbf{k}E + \boldsymbol{i}\, \mathbf{k}p_Z + \boldsymbol{j}\, m) & +RGB \\
(i\mathbf{k}E - \boldsymbol{i}\, \mathbf{i}p_X + \boldsymbol{j}\, m)\, (i\mathbf{k}E - \boldsymbol{i}\, \mathbf{j}p_Y + \boldsymbol{j}\, m)\, (i\mathbf{k}E - \boldsymbol{i}\, \mathbf{k}p_Z + \boldsymbol{j}\, m) & -RBG \\
(i\mathbf{k}E + \boldsymbol{i}\, \mathbf{i}p_X + \boldsymbol{j}\, m)\, (i\mathbf{k}E + \boldsymbol{i}\, \mathbf{j}p_Y + \boldsymbol{j}\, m)\, (i\mathbf{k}E + \boldsymbol{i}\, \mathbf{k}p_Z + \boldsymbol{j}\, m) & +BRG \\
(i\mathbf{k}E - \boldsymbol{i}\, \mathbf{i}p_X + \boldsymbol{j}\, m)\, (i\mathbf{k}E - \boldsymbol{i}\, \mathbf{j}p_Y + \boldsymbol{j}\, m)\, (i\mathbf{k}E - \boldsymbol{i}\, \mathbf{k}p_Z + \boldsymbol{j}\, m) & -GRB \\
(i\mathbf{k}E + \boldsymbol{i}\, \mathbf{i}p_X + \boldsymbol{j}\, m)\, (i\mathbf{k}E + \boldsymbol{i}\, \mathbf{j}p_Y + \boldsymbol{j}\, m)\, (i\mathbf{k}E + \boldsymbol{i}\, \mathbf{k}p_Z + \boldsymbol{j}\, m) & +GBR \\
(i\mathbf{k}E - \boldsymbol{i}\, \mathbf{i}p_X + \boldsymbol{j}\, m)\, (i\mathbf{k}E - \boldsymbol{i}\, \mathbf{j}p_Y + \boldsymbol{j}\, m)\, (i\mathbf{k}E - \boldsymbol{i}\, \mathbf{k}p_Z + \boldsymbol{j}\, m) & -BGR
\end{array}$$

This has exactly the same group structure as the standard 'coloured' baryon wavefunction made of R, G and B 'quarks',

$$\psi \sim (RGB - RBG + BRG - GRB + GBR - BGR)$$

That is, it has an $SU(3)$ structure, with 8 generators. And, since the E and **p** terms in the state vector really represent time and space derivatives, we can replace these with the covariant derivatives needed for invariance under a local $SU(3)$ gauge transformation.

A significant aspect of this $SU(3)$ symmetry or strong interaction is that it is entirely nonlocal. That is, the exchange of momentum **p** involved is entirely independent of any spatial position of the 3 components of the baryon. We can, therefore, suppose that the rate of change of momentum (or 'force') is constant with respect to spatial positioning or separation. A constant force is equivalent to a potential which is linear with distance, exactly as is required for the conventional strong interaction.

Very significantly, the full symmetry between the 3 momentum components can only apply if the momentum operators can be equally positive or negative. With all phases of the interaction present at the same time (perfect gauge invariance), this is equivalent to saying that left-handedness and right-handedness must be present simultaneously in the baryon state. In other words, the baryonic state must have non-zero mass via the Higgs mechanism. So this formalism provides an automatic solution of the mass-gap problem, and it ensures that the mass of baryons, even if calculated in QCD from the interactions of quarks and gluons, is generated ultimately by exactly the same mechanism as that of the W and Z bosons.

It is significant that the baryon representation can only exist as a unified or entangled state. It is not really a representation of a combination of 3 independent fermions. It is equally significant that the representation is impossible in a conventional spinor formulation, with terms such as $p_X + ip_Y$, or in any representation in which the momentum operators cannot show the full affine nature of the vector concept.

7 Multiple meanings for *i, j, k*

The important thing now is to see what physical information is *identifiable within the structure we have created*. The mathematics only becomes significant when it is associated with a physical application. We might expect, for instance, that the ***k***, ***i***, ***j*** operators are active physical elements ('hypertext') rather than passive mathematical objects. In fact, the three quaternion operators ***i***, ***j***, ***k*** can be seen to have multiple meanings in the nilpotent formalism – as charge generators; as *P*, *C*, *T* transformation operators; and as vacuum generators.

(1) The primary meaning is as charge generators.
(2) Premultiplying the nilpotent gives vacuum, e.g. $\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ acts as a weak vacuum.
(3) Pre- and postmultiplying the nilpotent transforms via *P*, *C* or *T* operations: e.g. $\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}$ becomes a *T* transformation.

If we take $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and postmultiply it by $\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, the result is $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, multiplied by a scalar, which can be normalized away. This can be done an indefinite number of times. That is, the idempotent $\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ behaves as a vacuum operator. The same applies to $\boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and $\boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$.

We can see the three vacuum coefficients ***k***, ***i***, ***j*** as originating in (or being responsible for) the concept of discrete (point-like) charge. The operators act as a discrete partitioning of the continuous vacuum responsible for zero-point energy. In this sense, they are related to weak, strong and electric localized charges, though they are delocalized. We can, in fact, suggest specific identifications on the basis of the pseudoscalar, vector and scalar characteristics of the associated terms.

$\boldsymbol{k}\,(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	weak vacuum	fermion creation
$\boldsymbol{i}\,(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	strong vacuum	gluon plasma
$\boldsymbol{j}\,(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	electric vacuum	*SU*(2)

The 3 additional terms in the Dirac spinor then become strong, weak and electric vacuum 'reflections' of the state defined by the lead term.

CPT symmetry uses exactly the same operators This is, of course, not a coincidence. If we define the three operations in nilpotent form by:

$$P \qquad \boldsymbol{i}\,(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\,\boldsymbol{i} = (i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$$

$$T \qquad \boldsymbol{k}\,(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\,\boldsymbol{k} = (-i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)$$

$$C \qquad -\boldsymbol{j}\,(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\,\boldsymbol{j} = (-i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$$

then the combined operation *CPT* becomes

$$-\boldsymbol{j}\,(\boldsymbol{i}\,(\boldsymbol{k}\,(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\,\boldsymbol{k})\,\boldsymbol{i})\,\boldsymbol{j} = (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)$$

Significantly, *CPT* is defined to connect relativity with causality, and this is only possible in nilpotent form, where the fifth term in the expression (rest mass or proper time) directly incorporates causality.

8 Interaction vertices and bosonic states

We have shown that two of the three fundamental interactions – the Coulomb or electric, and the strong – emerge from the nilpotent structure alone. While the first is a purely scalar term, the second derives from the vector properties of the **p** operator. We can now specify that the third fundamental interaction, the weak interaction, defined as a harmonic oscillator, creating and destroying of bosonic states, is also a property of the structure of the nilpotent operator. This time the structural element is the 4-component spinor, with its alternating sign terms for E and **p**. It can be shown, conventionally (and also in nilpotent terms), that the Dirac equation or Dirac operator automatically generates a *zitterbewegung*, or switching between the four terms by a mechanism which is equivalent to the weak interaction, producing bosonic vertices in which one fermionic state is created while another is destroyed. Although, within a single fermion structure, this interaction remains virtual because only one of the four states is real, it can be generalised to real interactions between real fermionic states.

Because the state vector always represents four terms with the complete variation of signs in E and **p**, an interaction vertex between any fermion / antifermion and any other

$$(i\boldsymbol{k}E_1 + \mathbf{ip}_1 + \boldsymbol{j}m_1)\,(i\boldsymbol{k}E_2 + \mathbf{ip}_2 + \boldsymbol{j}m_2)$$

will remove the quaternionic operators, leaving only scalars and vectors. When the E, **p** and m values become numerically equal, the vertex can be defined as a new *combined* bosonic state, with a single phase. We may note here that nilpotent wavefunctions or amplitudes are automatically antisymmetric:

$$(\pm\, i\boldsymbol{k}E_1 \pm \mathbf{ip}_1 + \boldsymbol{j}m_1)\,(\pm\, i\boldsymbol{k}E_2 \pm \mathbf{ip}_2 + \boldsymbol{j}m_2) - (\pm\, i\boldsymbol{k}E_2 \pm \mathbf{ip}_2 + \boldsymbol{j}m_2)\,(\pm\, i\boldsymbol{k}E_1 \pm \mathbf{ip}_1 + \boldsymbol{j}m_1)$$

$$= 4\mathbf{p}_1\mathbf{p}_2 - 4\mathbf{p}_2\mathbf{p}_1 = 8\, i\, \mathbf{p}_1 \times \mathbf{p}_2$$

Where there is an interaction vertex between two fermionic / antifermionic states, the signs of E and **p** of the second term, with respect to the first, will also determine the nature of the bosonic or combined state which may be created. Because there are three operators involved – $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ – there are also three possible bosonic states. Any transformation of a fermionic state can be represented as a bosonic state in which the old state is annihilated and the new one created (with a transformation equivalent to T, C or P).

Spin 1 boson:	$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (-\ i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)$	T
Spin 0 boson:	$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (-\ i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$	C
π Berry phase, etc.:	$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$	P

The spin 0 fermion-fermion state $(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)$ has many physical manifestations: Aharonov-Bohm effect in electrons; Jahn-Teller effect; quantum Hall effect; Cooper pairs (He^4); even-even nuclei; Bose-Einstein condensate. Even spin 1 He^3 can be accommodated if we regard it as constituted of physically separated components involved in a harmonic oscillator-type motion in opposite directions, and so with $+\ \boldsymbol{i}\mathbf{p}$ and $-\ \boldsymbol{i}\mathbf{p}$ adding up to spin 1.

Significantly, the spin 0 bosonic state cannot be massless, because, if it is nilpotent it automatically becomes zero:

$$(i\boldsymbol{k}E + \mathbf{ip})\ (-\ i\boldsymbol{k}E - \mathbf{ip}) = 0$$

This becomes a significant factor in the Higgs mechanism. It also implies that massless fermions cannot have the same handedness as massless antifermions. The conventional derivation of spin assigns left-handedness to fermions.

In the case of the strong force, the mediators will be six gluons of the form:

$$(i\boldsymbol{k}E - \mathbf{ii}p_x)\ (-\ i\boldsymbol{k}E - \mathbf{ij}p_y)$$

and two combinations of the three bosons of the form:

$$(i\boldsymbol{k}E - \mathbf{ii}p_x)\ (-\ i\boldsymbol{k}E - \mathbf{ii}p_x).$$

These structures are, of course, identical to an equivalent set in which both brackets undergo a complete sign reversal. The important thing here is that applying any of these mediators will produce a sign change in the **p** component that leads to mass.

9 Vacua and renormalization

We can see how the 3 bosonic states are related to vacua produced by the 3 charge operators:

weak spin 1

$$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{k}\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{k}\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{k}\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \ldots$$
$$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (-i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (-i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \ldots$$

electric spin 0

$$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{j}\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{j}\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \boldsymbol{j}\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ \ldots$$
$$(i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (-\ i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)\ (i\boldsymbol{k}E + \mathbf{ip} + \boldsymbol{j}m)\ (-\ i\boldsymbol{k}E - \mathbf{ip} + \boldsymbol{j}m)\ \ldots$$

strong B-E condensate

$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ \boldsymbol{i}\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ \boldsymbol{i}\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ \boldsymbol{i}\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots$

$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots$

All these discrete vacuum states produce virtual boson states which have no effect on the fermion ($\boldsymbol{ik}E$ + $\boldsymbol{i}\mathbf{p}$ + $\boldsymbol{j}m$). So, each fermion becomes *its own* supersymmetric bosonic partner, and vice versa. This removes the need for renormalization in the case of free particles, while 'renormalization' of interacting particles becomes rescaling – charge values being determined by their interactions with all the others in the universe.

We can show this nilpotent property relatively simply. The perturbation expansion for a first-order coupling of a virtual photon to an electron in a 4-potential (ϕ, $\mathbf{A}$) produces a wavefunction of the form:

$$\Psi_1 = -e\Sigma\ [\boldsymbol{k}E + i\boldsymbol{i}\ \boldsymbol{\sigma}.(\mathbf{p} + \mathbf{k}) + i\boldsymbol{j}m]^{-1}(i\boldsymbol{k}\phi - \boldsymbol{i}\ \boldsymbol{\sigma}.\mathbf{A})(\boldsymbol{k}E + i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)\ e^{-i(Et - \mathbf{p.r})}$$

Observing in the rest frame of the electron and eliminating any *external* source of potential (by assuming only self-potential), then the external momentum $\mathbf{k}$ = 0 and $\boldsymbol{i}\ \boldsymbol{\sigma}.\mathbf{A}$ = 0. So

$$\Psi_1 = -e\ \Sigma\ [\boldsymbol{k}E + i\boldsymbol{i}\ \boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m]^{-1}\ i\boldsymbol{k}\phi\ (\boldsymbol{k}E + i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)\ e^{-i(Et - \mathbf{p.r})} = 0.$$

Renormalization of the charge and mass of free fermions in the conventional formalism is a result of creating a 'redundancy barrier' through an artificial singularity. Another example of this is the *infrared divergence* in the definition of the propagator.

In the Feynman formalism, the fermion propagator has the form:

$$S_F(p) = \frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2}$$

with a singularity or 'pole' (p_0) where $p^2 - m^2 = 0$. This singularity goes all the way back to having to separate positive and negative energy solutions in the conventional Dirac formalism. Here, on either side of the singularity, we have positive energy states moving forwards in time, and negative energy states moving backwards in time, the terms. Using a contour integral method we arrive at:

$$S_F(x - x') = \int d^3p \frac{1}{(2\pi)^3} \frac{m}{2E}\left[-i\theta(t - t')\sum_{r=1}^{2}\Psi(x)\overline{\Psi}(x') + i\theta(t' - t)\sum_{r=3}^{4}\Psi(x)\overline{\Psi}(x')\right].$$

In the nilpotent formalism, the fermion propagator becomes:

$$S_F(p) = \frac{1}{\pm\boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m}\ ,$$

with no singularity or pole, because

$$\frac{1}{\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m} = \frac{\mp \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m}{(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)(\mp \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)} = \frac{\mp \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m}{E^2 + p^2 + m^2}$$

is finite at all values. There is *no infrared divergence* because there is no separation of positive and negative energy states, or forward and backward times. The integral now becomes:

$$S_F(x - x') = \int d^3 p \frac{1}{(2\pi)^3} \frac{m}{2E} \theta(t - t') \Psi(x) \overline{\Psi}(x') \ ,$$

in which

$$\Psi(x) = (\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m) \exp(ipx)$$

and the adjoint term

$$\overline{\Psi}(x') = (\pm \boldsymbol{k}E \mp i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} - i\boldsymbol{j}m)...) \ (i\boldsymbol{k}) \exp(-ipx')$$

are respective row and column vectors. The integral comes as a *single package*.

Three boson propagators can be defined by analogy with the fermion propagator.

Spin 1: $$\Delta_F(x - x') = \frac{1}{(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)(\mp \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)} \ ,$$

Spin 0: $$\Delta_F(x - x') = \frac{1}{(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)(\mp \boldsymbol{k}E \mp i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)} \ ,$$

and the fermion-fermion state, which represents the Bose-Einstein condensate / nonzero Berry phase:

$$\Delta_F(x - x') = \frac{1}{(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)(\pm \boldsymbol{k}E \mp i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p} + i\boldsymbol{j}m)} \ .$$

For QED, where the boson is massless, we have:

$$\Delta_F(x - x') = \frac{1}{(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p})(\mp \boldsymbol{k}E \pm i\boldsymbol{i}\boldsymbol{\sigma}.\mathbf{p})} \ .$$

In the specific case of massless bosons, conventional theory states that ‘infrared’ divergencies occur when such bosons are emitted from an initial or final stage which is on the mass shell. Such divergencies, however, will not occur where there is no pole.

10 The Higgs mechanism, Berry phase and strings

The nilpotent structures allow us to do a simple derivation of the principle of the Higgs mechanism, although a more detailed analysis requires using the symmetry groups relevant to the interactions involved. Let us imagine a virtual fermionic state with no mass in vacuum

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p})$$

An ideal vacuum would maintain exact and absolute *C*, *P* and *T* symmetries. Under *C* transformation, $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p})$ would become

$$(-\,\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p})$$

with which it would be indistinguishable under normalization. No bosonic state would be required for the transformation. If, however, the vacuum state is degenerate in some way under charge conjugation (as supposed in the weak interaction), then

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p})$$

will be transformable into a state which can be distinguished from it, and the bosonic state $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p})\ (-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p})$ will necessarily exist. However, this can only be true if the state has nonzero mass and becomes the spin 0 'Higgs boson':

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

The massiveness of the spin 0 state also has consequences for cases involving a nonzero Berry phase, and specifically those involving the phase value π. In these cases, the spin 0 'bosonic' state

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

is such as would be required in a pure weak transition from $-\boldsymbol{ik}E$ to $+\ \boldsymbol{ik}E$, or its inverse. Because the spin 0 state is necessarily massive, time reversal symmetry (the one applicable to the transition) must be broken in the weak formation or decay of states involving a nonzero Berry phase. A likely possibility for experimental investigation is the quantum Hall effect in grapheme.

The nilpotent structure also allows some observations regarding string theory. In this context, the nilpotent operator

$$(\pm\ \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

can be regarded as a 10-D object (embedded in Hilbert space). Five dimensions are required for iE, $\mathbf{p}$, m and five for $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$; and six of these (all but iE and $\mathbf{p}$) are compactified, in the sense that they are fixed or conserved quantities. It is immediately apparent that the nilpotent structure fulfils the criteria for a perfect string theory, as one in which 'self-duality in phase space determines vacuum selection'. In addition, it is a mass-shell system and incorporates the right groups. This does not imply that string theory is necessary, but that the virtues claimed for such theories are fully incorporated into the nilpotent structure, without requiring the arbitrary model-dependent aspects.

11 Conclusion

This paper gives only a few results that have been obtained by the nilpotent method. Further work shows that it can encompass a large amount of QM, QFT and the Standard Model, in a coherent structure, with some hints of what might lie beyond.[2,3] The key to its success lies in the fact that it avoids the conventional distortion of the structure of 4-D space-time through the unique Clifford algebra which unlocks the Dirac code.

References

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