How do you add relative velocities?[1]

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Abstract. The Lorentz boost entails relative velocity to be ternary: ternary relative velocity is a velocity of a body with respect to an interior observer *as seen* by a preferred exterior observer. The Lorentz boosts imply not associative addition of ternary relative velocities. New concepts are introduced: Lorentz-group-free binary relative velocities with associative addition, and the Lorentz-group-free boost. Observer-independence, and the Lorentz-invariance, are distinct concepts. This suggest the possibility of formulating many-body relativistic dynamics without Lorentz/Poincare invariance.

Contents

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1. The addition of Einstein's velocities is not associative

The following identity holds for three arbitrary vectors in arbitrary dimensions: for the Grassmann's wedge product and inner product acting as the graded derivation of the Grassmann algebra,

$$\mathbf{w}\cdot(\mathbf{v}\wedge\mathbf{u}) = (\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v} \qquad (= \mathbf{w}\times(\mathbf{u}\times\mathbf{v})). \tag{1.1}$$

On the right of (1.1) there is the double Gibbs's cross product of vectors that is orientation-dependent. However we prefer the orientation-independent Grassmann's exterior product, than the Gibbs's internal product, for two reasons. Firstly, in dimensions $\neq$ three, Gibbs's product needs the extra explication, like in [Plebański and Przanowski 1988], and, secondly, because of this superfluous orientation-dependence. The Gibbs's product will be not used in the present paper.

The Heaviside-FitzGerald-Lorentz scalar factor is denoted by $\gamma_{\mathbf{v}} \equiv \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}$.
Sometimes for simplicity of formulas the light velocity is set $\mathbf{c}^2 = 1$. In particular (1.1) gives

$$\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{u}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} - c^2\left(1-\frac{1}{\gamma_{\mathbf{u}}^2}\right)\mathbf{v}. \tag{1.2}$$

1.1 Definition (Isometric relative velocity). The space-like velocity $\mathbf{v}$ of a time-like body R relative to time-like Q is said to be *isometric*, or *ternary*, if it is defined in terms of the isometric Lorentz boost, $B_{P\wedge\mathbf{v}}Q = R$, where $B_{P\wedge\mathbf{v}} \in O(1,3)$.

The above definition is motivated by the following theorem. For the massive three-body system given in terms of the three time like vectors $\{P,Q,R\}$, the Lorentz-boost-link equation for unknown space-like velocity $\mathbf{v}$, $B_{P\wedge\mathbf{v}}Q = R$, has the unique solution, $\mathbf{v} = \mathbf{v}(P,Q,R)$, see Section 3. This *ternary* velocity-solution is reciprocal, $\mathbf{v}(P,Q,R) = -\mathbf{v}(P,R,Q)$.

The principal aim of the present note is to introduce the binary relative velocity-morphism that can *not* parameterize the isometric Lorentz transformation.

In 1905 Albert Einstein introduced relativity of simultaneity, and derived the addition of relative velocities parameterized the isometric Lorentz transformations. The $\oplus$-addition of such Einstein's isometric velocities reads as follows [Fock 1955, 1961 §16, formula (16.08)]

$$\mathbf{v}\oplus\mathbf{u} = \frac{\mathbf{u}+\mathbf{v}}{1+\mathbf{v}\cdot\mathbf{u}} \quad + \quad \frac{\gamma_{\mathbf{u}}}{(\gamma_{\mathbf{u}}+1)}\,\frac{\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{u})}{(1+\mathbf{v}\cdot\mathbf{u})} \tag{1.3}$$

$$= \frac{\gamma_{\mathbf{u}}\,\mathbf{u}+\mathbf{v}}{\gamma_{\mathbf{u}}(1+\mathbf{v}\cdot\mathbf{u})} + \frac{\gamma_{\mathbf{u}}}{(\gamma_{\mathbf{u}}+1)}\,\frac{(\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}}{(1+\mathbf{v}\cdot\mathbf{u})} \quad \Longrightarrow \quad \gamma_{\mathbf{v}\oplus\mathbf{u}} = \gamma_{\mathbf{v}}\,\gamma_{\mathbf{u}}(1-\mathbf{v}\cdot\mathbf{u}^{-1}). \tag{1.4}$$

The above two versions of the law of addition, (1.3) and (1.4), are related by the identity (1.1)-(1.2). The first version is convenient for the particular case of addition of collinear isometric relative velocities, *i.e.* for $\mathbf{v}\wedge\mathbf{u} = 0$. The second version (1.4) is convenient for the particular case of the addition of perpendicular isometric relative velocities, *i.e.* for $\mathbf{v}\cdot\mathbf{u} = 0$, when $\mathbf{v}\oplus\mathbf{u} = \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}$.

In the limit to absolute simultaneity (absolute = observer-free), $\oplus$-addition becomes the Galilean-Newtonian $(+)$-addition of relative velocities. Formally $(+)$-addition is an abelian group, and the set of all relative velocities become vectors of a linear algebra.

Here are three deficient properties of the $\oplus$-addition of isometric relative velocities (1.3)-(1.4) [Einstein 1905; Fock 1955, 1961].

(i) The $\oplus$-inverse is the reciprocal velocity, $\mathbf{u}^{-1} = -\mathbf{u}$, as in the case of absolute time:

$$\mathbf{v} \oplus \mathbf{u} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{v} + \mathbf{u} = \mathbf{0}, \quad \text{that is:} \quad \oplus - \text{inverse} \; = \; (+) - \text{inverse}. \tag{1.5}$$

(ii) The coincidence of the Galilean $(+)$-inverse and the Einstein $\oplus$-inverse, gives the Mocanu paradox [Mocanu 1985, 1986]:

$$(\mathbf{v} \oplus \mathbf{u})^{-1} = (\mathbf{v}^{-1}) \oplus (\mathbf{u}^{-1}) \quad \neq \quad (\mathbf{u}^{-1}) \oplus (\mathbf{v}^{-1}). \tag{1.6}$$

$\oplus$-inverse is $\oplus$-automorphism. Whereas one would expect that the unary inverse operation is an *anti*-automorphism, $(f \circ g)^{-1} = (g^{-1}) \circ (f^{-1})$.

(iii) In 1988 Ungar discovered that the $\oplus$-addition is not associative [Ungar 1988, p. 71]. Indeed, one can calculate for the two alternative bracketing:

$$\frac{\gamma_{\mathbf{w} \oplus (\mathbf{v} \oplus \mathbf{u})}}{\gamma_{\mathbf{w}} \gamma_{\mathbf{v}} \gamma_{\mathbf{u}}} = 1 + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u} + \frac{(\mathbf{w} \wedge \mathbf{u}) \cdot (\mathbf{u} \wedge \mathbf{v})}{1 + \sqrt{1 - \mathbf{u}^2}},$$
$$\frac{\gamma_{(\mathbf{w} \oplus \mathbf{v}) \oplus \mathbf{u}}}{\gamma_{\mathbf{w}} \gamma_{\mathbf{v}} \gamma_{\mathbf{u}}} = 1 + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u} + \frac{(\mathbf{w} \wedge \mathbf{v}) \cdot (\mathbf{v} \wedge \mathbf{u})}{1 + \sqrt{1 - \mathbf{v}^2}}, \tag{1.7}$$
$$\{\mathbf{w} \oplus (\mathbf{v} \oplus \mathbf{u})\} \wedge \{(\mathbf{w} \oplus \mathbf{v}) \oplus \mathbf{u}\} = A(\mathbf{w} \wedge \mathbf{v}) + B(\mathbf{v} \wedge \mathbf{u}) + C(\mathbf{u} \wedge \mathbf{w}) \neq 0. \tag{1.8}$$

Thus not only are these two resulting relative velocities not parallel (1.8), but also their differ in their scalar magnitude.

1.2 Note. An analysis of the derivation of the Lorentz group as the group of transformations relating observers, and the velocity $\oplus$-addition (1.3)-(1.4) in [Einstein 1905] reveals that the inverse property (1.5) is a tacit assumption used effectively as an axiom and is not derived from Einstein's two postulates. The reciprocity axiom (1.5) tells that every observer measuring some velocity can measure also inverse of this velocity. It is however true that the property (1.5) is necessary for the derivation of the Lorentz group as the one that relates two observers:

$$\{\oplus\text{-inverse} = (+)\text{-inverse}\} \quad \Longleftrightarrow \quad \text{Lorentz group relating observers.}$$

1.3 Note. We want to call attention to the fact that the reciprocity property (1.5) does not agree with the relativity of proper-time. Bodies mutually moved must possess different simultaneous relations, therefore the velocity $\mathbf{u}$ and his inverse $\mathbf{u}^{-1}$ are each tangent to a different instantaneous space and these two spaces are not parallel. Equivalently, $\mathbf{u}$ and $\mathbf{u}^{-1}$, are in the kernels of the different proper-time differential forms. It its true that $|\mathbf{u}^{-1}| = |\mathbf{u}|$, however an observer measuring a space-like velocity $\mathbf{u}$ can *not* see (spacetime direction of) *her/his* velocity relative to observed body. That is she/he can not measure (the spacetime direction of) the inverse velocity $\mathbf{u}^{-1}$.

1.4 Note. Being not associative, the Einstein addition is not a group operation. Not associative $\oplus$-addition is counterintuitive and paradoxical: for a system of four or more bodies the $\oplus$-addition of three not parallel velocities gives the **two** distinct velocities between two bodies. There have been attempts [Ungar 2001] to explain the not-associativity and Mocanu paradoxes as the Thomas rotation (Thomas in 1926), *i.e.* as not transitivity of the parallelism of the spatial frames. We consider this attempt not satisfactory. Jackson [1962] argued that the Thomas rotation is *necessary* in order to explain factor '2' in the doublet separation for spin-orbit interaction. Einstein was surprised that Thomas's relativistic 'correction' could give factor '2'.

Dirac in 1928 explained the same factor and the correct spin levels in terms of the Clifford algebra and the Dirac equation, without invoking the Thomas rotation. The Dirac equation conceptually ought to be understood in terms of the Clifford algebra alone. No longer did anyone need Thomas's precession except for the not associative $\oplus$-addition of velocities.

Herein we propose to formulate the physics of relativity in terms of the category of observers with binary relative velocities-morphisms that can not parameterize the isometric Lorentz transformations. These Lorentz-group-free binary relative velocities possess the associative $\circ$-addition, see Tables 1-2. This associative $\circ$-addition is a trivial corollary that follows from two related new concepts: the binary relative velocity is a categorical morphism, and a new, not isometric boost, that is Lorentz-group-free. In the consequence a bivector, $\mathbf{u}\wedge(\mathbf{u}^{-1})\neq 0$, does not vanish. The $\circ$-inverse velocity $\mathbf{u}^{-1}$ is given by an isometric Lorentz boost of the Galilean inverse $-\mathbf{u}$ [Matolcsi 1993 1.3.7, 1.3.8, 4.2.8; 2001 page 91; Bini et al. 1995 page 2551, formula (2.3)].

Table 1. How do you add relative velocities?

Galilean binary velocities	Einstein ternary velocities	categorical Lorentz-group-free binary velocities
Associative	Not associative	Associative
The absolute zero/neutral velocity observer-independent		Zero velocity is observer-dependent
The same absolute reciprocal inverse		Not Galilean inverse observer-dependent
Abelian group	Loop = unital quasigroup	Groupoid category Velocity is a morphism

Table 2. What it is the categorical, Lorentz-group-free relativity?

	Einstein's relativity imply the Lorentz **group**	Lorentz-group-free relativity is a groupoid **category**
Transformations among observers	Lorentz group	**Not** isometry
Addition of relative velocities	**Not** associative	Associative. Binary velocity is a morphism

2. Notation and terminology

In the following $\mathscr{F}$ denotes an $\mathbb{R}$-algebra of scalar fields on space-time, and $\operatorname{der}\mathscr{F}$ denotes a Lie $\mathscr{F}$-module of $\mathbb{R}$-derivations of a ring $\mathscr{F}$. Moreover g stands for a tensor field of Lorentzian metric with signature $(-+++)$, and will be considered as $\mathscr{F}$-module map:

$$(\operatorname{der}\mathscr{F}) \qquad \xrightarrow{g^*=g} \qquad (\operatorname{der}\mathscr{F})^*. \tag{2.1}$$

The names, ‘velocity’ and ‘relative velocity’, are used exclusively for bounded *space*-like vector fields $\in \operatorname{der}\mathscr{F}$. A set/category of all velocities is denoted by

$$\mathbb{V} \equiv \{\mathbf{v} \in \operatorname{der}\mathscr{F} | 0 \leq \mathbf{v}^2 < c^2\}. \tag{2.2}$$

All space-like velocities are denoted by lowercase bold roman characters $\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots \in \mathbb{V}$. The assumption that exists the finite limiting velocity, *i.e.* $(\gamma_{\mathbf{v}})^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right) \equiv 1$, can be *derived* as the corollary of a categorical approach to the velocity as a categorical morphism.

The terms *observer, observed, body, laboratory,* are used here as synonymous and exclusively for the time-like future-directed and normalized vector fields $\in \operatorname{der}\mathscr{F}$. The set/category of all observers/observed is denoted by $\boldsymbol{\varpi}$, $\operatorname{obj}\boldsymbol{\varpi} \equiv \{P \in \operatorname{der}\mathscr{F} | P^2 = -1 \in \mathscr{F}\}$. Objects of this category, that here are synonyms for massive bodies and particles, are denoted by uppercase letters $P, Q, R, S, \ldots \in \operatorname{obj}\boldsymbol{\varpi}$. The main subject of this note is addition of the *space*-like *velocities*. For this reason a phrase ‘4-velocity’, a synonym for our observer and observed $\in \operatorname{obj}\boldsymbol{\varpi}$, will be avoided as confusing.

For an observer P and a velocity $\mathbf{v}$, the condition $P \cdot \mathbf{v} = 0$ is interpreted as necessary and sufficient for observing $\mathbf{v}$ by P. In the Einstein-Fock formula (1.4) it is understood implicitly that the velocities $\mathbf{u}$ and $\mathbf{v}$ are space-like and can be measured by time-like preferred observer $P \in \operatorname{der}\mathscr{F}$, $P^2 = -1$, who is orthogonal to them, $P \cdot \mathbf{u} = P \cdot \mathbf{v} = 0$.

Let a space-like velocity $\mathbf{u} \in \mathbb{V}$, be a velocity of a body Q relative to an observer P. Then we display this velocity $\mathbf{u}$ as an actual categorical arrow (morphism) which starts/outgoes at observer P (P is a node of the directed graph), and ends/ingoes at an observed body Q,

$$\cdots \longrightarrow \quad P \quad \underset{\mathbf{u}^{-1}}{\overset{\mathbf{u}}{\rightleftarrows}} \quad Q \quad \longrightarrow \ldots; \qquad \left(Q = P \quad \Longleftrightarrow \quad \mathbf{u} = \mathbf{0} = \mathbf{u}^{-1}\right), \tag{2.3}$$

$$\begin{aligned} &\text{observer of } \mathbf{u} = P, && \text{observed body with } \mathbf{u} = Q, \\ &\text{observed body with } \mathbf{u}^{-1} = P, && \text{observer of } \mathbf{u}^{-1} = Q. \end{aligned}$$

3. Lorentz boost needs preferred exterior observer

The group of rotations $O(3)$ is not normal subgroup of the Lorentz group $O(3,1)$. Therefore there is no natural decomposition of the Lorentz transformation as a composition of a rotation and a boost. Every such decomposition, $O(3,1) \ni L = \text{Rotation} \circ \text{Boost}$, depends on an auxiliary choice of a preferred time-like observer P.

Let P be an observer and $\mathbf{v}$ be a space-like velocity such that $P \cdot \mathbf{v} = 0$. Let us define the $\mathscr{F}$-module endomorphisms $B_{P \wedge \overline{\mathbf{v}}} \in \operatorname{End}_{\mathscr{F}}(\operatorname{der}\mathscr{F})$ as the polynomial in the following trace-less operator $M_{P\wedge\overline{\mathbf{v}}}$,

$$\overline{\mathbf{v}} \equiv \gamma_{\mathbf{v}} \frac{\mathbf{v}}{c}, \quad M_{P\wedge\overline{\mathbf{v}}} \equiv P \otimes_{\mathscr{F}} (g\overline{\mathbf{v}}) - \overline{\mathbf{v}} \otimes_{\mathscr{F}} (gP) \quad \in \operatorname{End}_{\mathscr{F}}(\operatorname{der}\mathscr{F}), \tag{3.1}$$

$$B_{P\wedge\overline{\mathbf{v}}} \equiv \mathrm{id} + M_{P\wedge\overline{\mathbf{v}}} + \frac{\left(M_{P\wedge\overline{\mathbf{v}}}\right)^2}{\gamma_{\mathbf{v}}+1} \tag{3.2}$$

$$\Longrightarrow \quad B_{P\wedge\overline{\mathbf{v}}} \circ B_{-P\wedge\overline{\mathbf{v}}} = \mathrm{id} = B_{-P\wedge\overline{\mathbf{v}}} \circ B_{P\wedge\overline{\mathbf{v}}}, \quad B_{P\wedge\overline{\mathbf{v}}} P = \gamma_{\mathbf{v}} \left(\frac{\mathbf{v}}{c} + P\right). \tag{3.3}$$

The endomorphism $B_{P\wedge\overline{\mathbf{v}}}$ leaves invariant the space-like P-dependent 2-plane (no rotation!). Moreover $B_{P\wedge\overline{\mathbf{v}}}$ is a g-isometry, $B_{P\wedge\overline{\mathbf{v}}} \in O_g = O(3,1)$. Thus an endomorphism $B_{P\wedge\overline{\mathbf{v}}}$ is a Lorentz P-boost.

For a given Lorentz transformation $L \in O_g$ and a given preferred (exterior) observer P, a Lorentz P-boost $B_{P\wedge PL}$ is given by (3.3) where $M_{P\wedge\overline{\mathbf{v}}}$ must be replaced by

$$M_{P\wedge LP} \equiv P \otimes gLP - (LP) \otimes gP, \tag{3.4}$$

and the scalar Lorentz factor $\gamma_{\mathbf{v}}$ must be replaced by $-(LP)\cdot P$. Then a P-rotation is given by $R_L^P \equiv (B_{P\wedge LP})^{-1} \circ L$. One can check that $R_L^P P = P$.

The above P-decomposition of the Lorentz transformation $L \in O_g$, as a composition of the P-rotation and a P-boost, $L = B_{P\wedge LP} \circ R_L^P$, one can apply for the composition of two Lorentz P-boosts

$$B_{P\wedge\overline{\mathbf{u}}} \circ B_{P\wedge\overline{\mathbf{v}}} = B_{P\wedge\overline{\mathbf{v}\oplus_P\mathbf{u}}} \circ R^P(\mathbf{u},\mathbf{v}) \quad \in SO(1,3). \tag{3.5}$$

The composition of the Lorentz P-boosts is not a Lorentz P-boost, it is a P-boost *up* to the Thomas/Wigner P-rotation $R^P(\mathbf{u},\mathbf{v}) \in SO(3)$.[3]

The discussion of Lorentz boost commonly suppresses the observer-dependence, suggesting incorrectly that the Lorentz P-boost $B_{P\wedge\overline{\mathbf{v}}}$ is completely fixed by 'a velocity parameter $\mathbf{v}$', e.g. [Jackson 1962 §11, Ungar 2001 p. 254].

4. Lorentz-group-free, categorical boost

In this Section we define observer-independent boost $b_{\mathbf{u}}$ that appears to be not a g-isometry.

4.1 Axiom (Binary relative velocity). Let $P\cdot\mathbf{u} = 0$, $P^2 = -1$ and $\mathbf{u}^2 < c^2$. Then, and *only* then, $\exists!$ body $Q = b_{\mathbf{u}}P \equiv \gamma_{\mathbf{u}}(\mathbf{u}+P)$ moving with a velocity $\mathbf{u}$ *relative* to P. The space-like relative velocity $\mathbf{u}$ is said to be *binary*, $\mathbf{u} \equiv \varpi(P,Q) = \frac{Q}{\gamma_{\mathbf{u}}} - P$, $\gamma_{\mathbf{u}} = -Q\cdot P$.

Binary velocity is not reciprocal (inverse depends on internal observer),

$$\varpi(P,K) \equiv \frac{K}{-P\cdot K} - P \neq \varpi(K,P), \tag{4.1}$$

and therefore such velocity can not parameterize the isometry where $\{B(\mathbf{v})\}^{-1} = B(-\mathbf{v}) \in O(1,3)$.

[3] Added after publishing. In published text the definition of the $\oplus$-addition (3.5) were printed with incorrect order. This misprint is corrected here.

Clearly $(b_{\mathbf{u}}P)^2 = P^2 = -1$. Above Axiom implies that, $Q \cdot \mathbf{u} = \gamma_{\mathbf{u}} - \frac{1}{\gamma_{\mathbf{u}}}$. This Axiom motivates the following two diagrammatical rules for outgoing and ingoing arrows/velocities,

$$P \xrightarrow{\mathbf{u}} \ldots \qquad \text{'out' if and only if} \quad P \cdot \mathbf{u} = 0, \tag{4.2}$$

$$\ldots \xrightarrow{\mathbf{u}} Q \qquad \text{'in' if and only if} \quad Q \cdot \mathbf{u} = \gamma_{\mathbf{u}} - \frac{1}{\gamma_{\mathbf{u}}} \equiv \frac{\mathbf{u}^2}{\sqrt{1-\mathbf{u}^2}}. \tag{4.3}$$

A body Q can possess an ingoing velocity $\mathbf{u}$ if and only if $Q \cdot \mathbf{u} = \gamma_{\mathbf{u}} - \frac{1}{\gamma_{\mathbf{u}}}$. Then, and only then, the unique laboratory $P = (b_{\mathbf{u}})^{-1} Q = \frac{Q}{\gamma_{\mathbf{u}}} - \mathbf{u}$, exists, such that a body Q is moving with a velocity $\mathbf{u}$ *relative* to P.

In contrast to the Lorentz boost $B_{P \wedge \overline{\mathbf{u}}}$, (3.3), whose domain is the entire $\mathscr{F}$-module der$\mathscr{F}$, the domain of this *new* categorical boost $b_{\mathbf{u}}$ is not-linear two-dimensional sub-manifold of time-like normalized massive bodies that actually can measure the given space-like velocity $\mathbf{u} \in \mathbb{V}$,

$$\begin{gathered} \operatorname{domain}\{B_{P \wedge \mathbf{v}}\} = \operatorname{der}\mathscr{F}, \\ \operatorname{domain}\{b_{\mathbf{u}}\} = \{X \in \operatorname{der}]\mathscr{F} | X^2 = -1, X \cdot \mathbf{u} = 0\}. \end{gathered} \tag{4.4}$$

This exterior-observer-independent categorical boost b coincide with the Lorentz P-boost B_P (3.3) when acting on preferred observer only

$$Q \cdot \mathbf{u} = P \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad b_{\mathbf{u}} Q = \begin{cases} B_{P \wedge \overline{\mathbf{u}}} Q = \gamma_{\mathbf{u}}(\mathbf{u} + Q), & \text{iff } Q = P, \\ \neq B_{P \wedge \overline{\mathbf{u}}} Q, & \text{iff } Q \neq P. \end{cases} \tag{4.5}$$

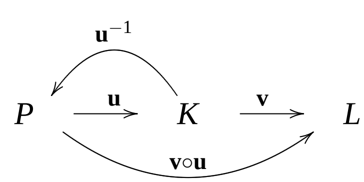
An observer-independent boost $b_{\mathbf{u}}$ is not g-isometry. Let $Q^2 = R^2 = -1$, $Q \cdot \mathbf{u} = R \cdot \mathbf{u} = 0$, and $\gamma_R^Q \equiv -Q \cdot R$. Then $(b_{\mathbf{u}}Q) \cdot (b_{\mathbf{u}}R) - Q \cdot R = -(\gamma_R^Q - 1)(\gamma_{\mathbf{u}}^2 - 1)$. The observer-independent categorical boost $b_{\mathbf{u}}$, is said also to be Lorentz-group-free.

The relativity theory with non-isometric boosts $\{b_{\mathbf{u}}\}$, do not violate the Lorentz invariance or covariance. The concept of Lorentz invariance is not applicable.

5. Associative addition of velocities

Consider a system of three bodies P, K and L, Figure 1. A body L is moving with a velocity $\mathbf{v}$ relative to a body K, and K is moving with a velocity $\mathbf{u}$ relative to an observer P. What is the velocity of L relative to P?

Figure 1. Three bodies $\{P, K, L\}$ in relative motions



We abbreviate 'the Lorentz-group-free addition of velocities' (which appears to be associative) to ∘-addition. One way to introduce ∘-addition is to consider the Lorentz-group-free boost b as an isomorphism from the composition of velocities-morphisms to composition of boosts/maps,

$$O(3,1) \not\ni \quad b_{\mathbf{v}\circ\mathbf{u}} \equiv b_{\mathbf{v}} \circ b_{\mathbf{u}} \qquad \text{whereas} \qquad B_{P\wedge\overline{\mathbf{v}\oplus_P\mathbf{u}}} \neq B_{P\wedge\overline{\mathbf{u}}} \circ B_{P\wedge\overline{\mathbf{v}}} \quad \in O(3,1), \tag{5.1}$$

$$\begin{aligned} P\cdot\mathbf{u}=0 \quad &\Longrightarrow \quad b_{\mathbf{u}}P = \gamma_{\mathbf{u}}(\mathbf{u}+P),\\ P\cdot\mathbf{u}=0,\ (b_{\mathbf{u}}P)\cdot\mathbf{v}=0 \quad &\Longrightarrow \quad b_{\mathbf{v}}(b_{\mathbf{u}}P) = \gamma_{\mathbf{v}}(\mathbf{v}+\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{u}}P),\\ P\cdot(\mathbf{v}\circ\mathbf{u})=0 \quad &\Longrightarrow \quad b_{\mathbf{v}\circ\mathbf{u}}P = \gamma_{\mathbf{v}\circ\mathbf{u}}(\mathbf{v}\circ\mathbf{u}+P),\\ \Longrightarrow \quad \mathbf{v}\circ_P\mathbf{u} &= \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}\circ\mathbf{u}}}(\mathbf{v}+\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{u}}P)-P. \end{aligned} \tag{5.2}$$

The scalar product of the vector P with formula (5.2), and jointly with Lemma 5.3 below, gives

$$\gamma_{\mathbf{v}\circ\mathbf{u}} = \gamma_{\mathbf{v}}\left(\gamma_{\mathbf{u}}+\frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right) = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}}\left(1-\frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right), \tag{5.3}$$

$$\mathbf{c}^2=1 \quad \Longrightarrow \quad \mathbf{v}\oplus\mathbf{c}=\mathbf{v}\circ\mathbf{c}=\mathbf{c} \quad \text{and} \quad (\mathbf{c}\oplus\mathbf{v})^2=(\mathbf{c}\circ\mathbf{v})^2=1. \tag{5.4}$$

Note that the space-like vectors possess the Euclidean angle if and only if they have the same time-like source. For example, see Figure 1, K is the source for $\mathbf{v}$ and $\mathbf{u}^{-1}$, $K\cdot\mathbf{v}=0=K\cdot\mathbf{u}^{-1}$. In Figure 1 and in Table 3, $P\cdot\mathbf{v}=-\mathbf{v}\cdot\mathbf{u}=\gamma_{\mathbf{u}}\mathbf{v}\cdot\mathbf{u}^{-1}$, see Lemma 5.3 below.

Table 3. The associative $\circ$-addition versus not associative $\oplus$-addition. The addition of orthogonal relative velocities, $\mathbf{v}\cdot\mathbf{u}^{-1}=0$, looks 'the same' for binary and ternary relative velocities.

$P\cdot\mathbf{u}=0, \quad K\cdot\mathbf{v}=0, \quad \mathbf{u}^{-1}\neq-\mathbf{u}, \quad \mathbf{v}^{-1}\neq-\mathbf{v} \quad \Longrightarrow$
$(1-\frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2})\,\mathbf{v}\circ\mathbf{u} = \mathbf{u}+\frac{\mathbf{v}}{\gamma_{\mathbf{u}}} + \frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c}P$
$P\cdot\mathbf{u}=0, \quad P\cdot\mathbf{v}=0, \quad \mathbf{u}^{-1}=-\mathbf{u}, \quad \mathbf{v}^{-1}=-\mathbf{v} \quad \Longrightarrow$
$(1-\frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2})\,\mathbf{v}\oplus_P\mathbf{u} = \mathbf{u}+\frac{\mathbf{v}}{\gamma_{\mathbf{u}}} - \frac{\gamma_{\mathbf{u}}}{(\gamma_{\mathbf{u}}+1)}\frac{(\mathbf{v}\cdot\mathbf{u}^{-1})}{c^2}\,\mathbf{u}$

For comparison the second line in Table 3 is again non-associative $\oplus$-addition (1.4), which can be presented in the form analogous to (5.2),

$$\gamma_{\mathbf{v}\oplus\mathbf{u}}\,\mathbf{v}\oplus\mathbf{u} = \gamma_{\mathbf{v}}\mathbf{v}+(\gamma_{\mathbf{v}\oplus\mathbf{u}}+\gamma_{\mathbf{v}})\frac{\gamma_{\mathbf{u}}\mathbf{u}}{\gamma_{\mathbf{u}}+1}. \tag{5.5}$$

5.1 Warning. Reader must not be misleading by shorter notation in Tables 3-4. In the first row for associative $\circ$-addition all relative velocities are binary, *i.e.* $\mathbf{v}\equiv\varpi(K,L)$ and $\mathbf{u}\equiv\varpi(P,K)$, as is exactly shown on Figure 1. Contrary to this, in the second row for the non-associative $\oplus$-addition, the same letters, $\mathbf{u}$ and $\mathbf{v}$, denotes the isometric ternary relative velocities, *i.e.* there $\mathbf{u}=\mathbf{u}(S,P,K)$ and $\mathbf{v}=\mathbf{v}(S,K,L)$, where the preferred exterior time-like observer S can be chosen arbitrarily, and this exterior observer S is not shown on Figure 1. Please note that the non-associative $\oplus$-addition expression is independent of the choice of the exterior observer S, the same formula holds if instead of $P\cdot\mathbf{u}=0=P\cdot\mathbf{v}$, we will assume that $S\cdot\mathbf{u}=0=S\cdot\mathbf{v}$, for completely arbitrary exterior observer S. Even if we made the particular choice $S=P$, as was done in the second rows in Tables 3-4, this does not means that the letter $\mathbf{u}$ in the first row and in the second row denotes exactly the same physical relative velocity, because the inverse is different. The binary velocity $\mathbf{u}$ in the first row of Tables 3-4 is *not* skew-symmetric function of his arguments, $\mathbf{u}=\varpi(P,K)\neq-\varpi(K,P)$. Whereas the ternary isometric velocity in the second row means always the reciprocal velocity, $\mathbf{u}(S,P,K)=-\mathbf{u}(S,K,P)$, and this must hold also for $S=P$, $\mathbf{u}(P,P,K)=-\mathbf{u}(P,K,P)$. Therefore conceptually, the Einstein's relative velocity parameterizing the isometric Lorentz boost is not the same as the binary relative velocity-morphism, $\mathbf{u}(P,P,K)\neq\varpi(P,K)$, even if numerically these expressions sometimes coincide.

All this means that the notation for the Heaviside-FitzGerald-Lorentz scalar factor, $\gamma_{\mathbf{u}}$, must not be identified in both rows in Tables 3-4. In the first rows $\gamma_{\mathbf{u}}\equiv-P\cdot K$, whereas in the second rows this factor depends on exterior observer,

$$\gamma_{\text{ternary}\,\mathbf{u}}=\gamma_{\mathbf{u}}(S,P,K)=\frac{(S\cdot P)^2+(S\cdot K)^2-P\cdot K-1}{2(S\cdot P)(S\cdot K)+P\cdot K+1}\neq-P\cdot K. \tag{5.6}$$

Elsewhere we showed Theorem that the magnitudes of the binary and ternary relative velocities coincide, $\gamma(\text{binary})=-P\cdot K=\gamma(\text{ternary})$, if and only if $(S\wedge P\wedge K)^2=0$.

The associative $\circ$-addition of relative binary velocities appears in Thesis [Świerk 1988]. Matolcsi [1993, §4.3], and Bini et al. [1995], derived the addition of relative binary velocities without observing the associativity, and without comparing with addition of Einstein's ternary reciprocal velocities (1.3)-(1.4). The Matolcsi's form need the following substitution into expression for the composition $\mathbf{v}\circ_P\mathbf{u}$,

$$\text{if}\quad \mathbf{u}\neq 0\qquad\text{then}\qquad P=-\frac{c}{\mathbf{u}^2}\left(\mathbf{u}+\frac{\mathbf{u}^{-1}}{\gamma_{\mathbf{u}}}\right). \tag{5.7}$$

5.2 Proposition (Inverse velocity). *A category of massive bodies is a groupoid category and therefore every body has his own separate zero velocity, i.e.* $\mathbf{v}$ *and* $\mathbf{v}^{-1}$ *'do not commute',*

$$\mathbf{0}_{observed}=\mathbf{v}\circ\mathbf{v}^{-1}\quad\neq\quad\mathbf{v}^{-1}\circ\mathbf{v}=\mathbf{0}_{observer}.$$

The $\circ$-inverse of the binary relative velocity possess the following properties:

$$\mathbf{v}^{-1}=-B_{P\wedge\bar{\mathbf{v}}}\mathbf{v}\quad\Longrightarrow\quad|\mathbf{v}^{-1}|=|\mathbf{v}|,\qquad\mathbf{v}^{-1}\cdot\mathbf{v}=-\frac{\mathbf{v}^2}{\sqrt{1-\mathbf{v}^2}}, \tag{5.8}$$

$$(\mathbf{v}+\mathbf{v}^{-1})^2=-2(\gamma_{\mathbf{v}}+1)\left(1-\tfrac{1}{\gamma_{\mathbf{v}}}\right)^2\approx\begin{cases}-(\mathbf{v})^4 & \textit{for}\quad|\mathbf{v}|<<1,\\ -\,2\gamma_{\mathbf{v}} & \textit{for}\quad|\mathbf{v}|\to 1.\end{cases} \tag{5.9}$$

Proof. Matolcsi observed the equality $\mathbf{v}^{-1}=\{\mathbf{v}(K,L)\}^{-1}=\mathbf{v}(L,K)=-B_{K\wedge L}\mathbf{v}(K,L)$, where the particular Lorentz boost is parameterized in terms of the initial K, and the final L, time-like vectors only [Matolcsi 1993 1.3.7, 1.3.8, 4.2.3; 2001 page 91]. See also [Bini, Carini and Jantzen 1995 formula (2.3)]. It was first admitted by Fahnline [1982, formulas (15)-(16)-(18)] that such 'binary' boost is not unique. The most general 'ternary' boost as the complete solution of the Lorentz-boost-problem was given in [Oziewicz 2005, unpublished]. We have

$$K\wedge L=K\wedge\bar{\mathbf{v}},\quad\text{and}\quad\mathbf{v}^{-1}=-B_{K\wedge\bar{\mathbf{v}}}\mathbf{v}=-\gamma\mathbf{v}-c\left(\gamma-\frac{1}{\gamma}\right)K.\quad\square \tag{5.10}$$

5.3 Lemma (Scalar identity for three-body-system). *Consider three-body-system in* Figure 1, *with binary relative velocities. Then,* $\mathbf{v}\cdot\mathbf{u} = -\gamma_{\mathbf{u}}\,\mathbf{v}\cdot\mathbf{u}^{-1}$. *Hence,* $\gamma_{\mathbf{v}\circ\mathbf{u}} = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}}(1-\mathbf{v}\cdot\mathbf{u}^{-1})$.

Proof. By Proposition 5.2, $\mathbf{u}^{-1} = -B_{P\wedge\overline{\mathbf{u}}}\,\mathbf{u}$. Definition (3.2) or (5.10), gives

$$B_{P\wedge\overline{\mathbf{u}}}\,\mathbf{u} \quad = \quad \gamma_{\mathbf{u}}\,\mathbf{u} + c\frac{\gamma_{\mathbf{u}}^2-1}{\gamma_{\mathbf{u}}}\,P. \tag{5.11}$$

Moreover, $\mathbf{v}\cdot K = 0$, imply that, $\mathbf{v}\cdot P = -\mathbf{v}\cdot\mathbf{u}$. All together leads to, $\mathbf{v}\cdot\mathbf{u}^{-1} = -\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}\cdot\mathbf{u}$. □

6. Three body system: collinear motion without Lorentz transformations

6.1 Lemma (Three body system). *Consider three body massive system,* Figure 1, *with binary relative velocities only (no exterior observer). One can verify the following circularly-permuted identities,*

$$\begin{aligned}
\mathbf{u}\wedge\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u}) &= \mathbf{u}^{-1}\wedge\mathbf{v}\wedge(\mathbf{v}\circ\mathbf{u})^{-1},\\
\mathbf{u}\wedge\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u})^{-1} &= \mathbf{u}^{-1}\wedge\mathbf{v}\wedge(\mathbf{v}\circ\mathbf{u}),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
\gamma_{\mathbf{v}\circ\mathbf{u}}\quad \tfrac{1}{c}(\mathbf{v}\circ\mathbf{u})\wedge\mathbf{u}\wedge\mathbf{v} &= (\mathbf{v}\cdot\mathbf{u}^{-1})\quad P\wedge K\wedge L\,,\\
\gamma_{\mathbf{v}}\quad \tfrac{1}{c}\mathbf{v}\wedge(\mathbf{v}\circ\mathbf{u})\wedge\mathbf{u}^{-1} &= (\mathbf{u}\cdot(\mathbf{v}\circ\mathbf{u}))\quad P\wedge K\wedge L\,,\\
\gamma_{\mathbf{u}}\quad \tfrac{1}{c}\mathbf{u}\wedge\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u}) &= (\mathbf{v}^{-1}\cdot(\mathbf{v}\circ\mathbf{u})^{-1})\quad P\wedge K\wedge L\,.
\end{aligned} \tag{6.2}$$

6.2 Definition (Collinear motion for binary velocities-morphisms). A massive three body system $\{P,K,L\}$, Figure 1, is said to be in collinear motion if and only if $P\wedge K\wedge L = 0$.

The above Definition 6.2 has the following motivation. To be the true three body system we must assume that $P\wedge K\neq 0$, $K\wedge L\neq 0$ and $L\wedge P\neq 0$. We must treat every massive body on Figure 1, with binary relative velocities only (no exterior observer), on equal footing. A priori collinearity seen by each body separately amounts to different condition.

Seen by P**:** the motion is collinear if and only if, $\mathbf{u}\wedge(\mathbf{v}\circ\mathbf{u}) = 0$.

Seen by K**:** the motion is collinear if and only if the relative velocity $\mathbf{v}$ and inverse $\mathbf{u}^{-1}$ seen by K, are collinear, $\mathbf{v}\wedge\mathbf{u}^{-1} = 0$.

Seen by L**:** the motion is collinear if and only if, $\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u})^{-1} = 0$.

Lemma 6.1 tells that *all* three the above conditions jointly imply necessarily the vanishing of the tri-vector $P\wedge K\wedge L = 0$. One can ask: does exists the collinear motion for $P\wedge K\wedge L\neq 0$? Supposing that $P\wedge K\wedge L\neq 0$ and the collinear motion as seen by any of these bodies separately, we will arrive to contradiction that at least one of the bi-vectors, $\{P\wedge K, K\wedge L, L\wedge P\}$, must vanish, *i.e.* the contradiction. Therefore the Definition 6.2 is the only possibility.

One can shows conversely that the Definition 6.2 imply the collinear motion for each observer separately. Namely

$$P\wedge K\wedge L = 0 \quad\Longrightarrow\quad (\gamma_{\mathbf{u}}^2-1)L = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}}\frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}P + \gamma_{\mathbf{v}}\left(\gamma_{\mathbf{u}}^2 - 1 - \gamma_{\mathbf{u}}^2\frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right)K \tag{6.3}$$

$$\Longrightarrow\quad \mathbf{v}\wedge\mathbf{u}^{-1} = 0,\quad \text{etc.} \tag{6.4}$$

For the case of non-collinear motion, the composed velocity $\mathbf{v} \circ \mathbf{u}$, needs to be the linear combination of *three* binary velocities, $\mathbf{u}, \mathbf{v}$ and inverse $\mathbf{u}^{-1}$ [Świerk 1988, Matolcsi 1993]. In the case of the collinear motion, the composed velocity $\mathbf{v} \circ \mathbf{u}$, must be linear combination of two velocities, $\{\mathbf{u}, \mathbf{v}\}$ or $\{\mathbf{u}, \mathbf{v}^{-1}\}$ or $\{\mathbf{v}, \mathbf{u}^{-1}\}$, only. For example, for some scalars, a and b, $\mathbf{v} \circ \mathbf{u} = a\mathbf{u} + b\mathbf{v}$. However $P \cdot \mathbf{v} = \gamma_{\mathbf{u}} \mathbf{v} \cdot \mathbf{u}^{-1} \neq 0$, and $b = 0$. Therefore $\mathbf{v} \circ \mathbf{u} = a\mathbf{u}$ and

$$c^2(\gamma_{\mathbf{u}}^2 - 1)\, \mathbf{v} \circ \mathbf{u} \quad = \quad \gamma_{\mathbf{u}}^2 (\mathbf{u} \cdot (\mathbf{v} \circ \mathbf{u}))\, \mathbf{u}. \tag{6.5}$$

This addition of collinear *binary* velocities 6.5 looks different from addition of collinear *ternary* velocities (1.3) [Einstein 1905],

$$\text{if} \qquad \mathbf{v} \wedge \mathbf{u} = 0 \qquad \text{then} \qquad \mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}}. \tag{6.6}$$

One can arrive to more explicit form of addition using identity (1.1) for $\mathbf{u}^{-1} \cdot (\mathbf{v} \wedge \mathbf{u}^{-1})$, with substitution (5.7),

$$(\gamma_{\mathbf{u}}^2 - 1) \left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) \mathbf{v} \circ \mathbf{u} = \left(\gamma_{\mathbf{u}}^2 - 1 - \gamma_{\mathbf{u}}^2 \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) \mathbf{u} - \gamma_{\mathbf{u}} \mathbf{u}^{-1} \cdot \frac{(\mathbf{v} \wedge \mathbf{u}^{-1})}{c^2}. \tag{6.7}$$

In particular for collinear binary velocities $\mathbf{v} \wedge \mathbf{u}^{-1} = 0$, the last term in (6.7) vanishes, and we have

$$\mathbf{v} \circ \mathbf{u} = \left\{ \frac{1 - \frac{\gamma_{\mathbf{u}}^2}{\gamma_{\mathbf{u}}^2 - 1} \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}}{1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}} \right\} \mathbf{u}, \quad \text{where} \quad \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2} = \frac{\sqrt{(\gamma_{\mathbf{u}}^2 - 1)(\gamma_{\mathbf{v}}^2 - 1)}}{\gamma_{\mathbf{u}} \gamma_{\mathbf{v}}}, \tag{6.8}$$

$$\lim_{\gamma_{\mathbf{u}} \longrightarrow \infty} (\mathbf{v} \circ \mathbf{u}) = \mathbf{u}. \tag{6.9}$$

Table 4. The associative $\circ$-addition versus not associative $\oplus$-addition. The addition of collinear relative velocities, $\mathbf{v} \wedge \mathbf{u}^{-1} = 0$, looks different for binary and ternary relative velocities.

$P \cdot \mathbf{u} = 0, \quad K \cdot \mathbf{v} = 0, \quad \mathbf{u}^{-1} \neq -\mathbf{u}, \quad \mathbf{v}^{-1} \neq -\mathbf{v} \quad \Longrightarrow$ $(\gamma_{\mathbf{u}}^2 - 1)\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) \mathbf{v} \circ \mathbf{u} \quad = \quad \left\{\gamma_{\mathbf{u}}^2 \left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) - 1\right\} \mathbf{u} - \gamma_{\mathbf{u}} \mathbf{u}^{-1} \cdot \frac{(\mathbf{v} \wedge \mathbf{u}^{-1})}{c^2}$
$P \cdot \mathbf{u} = 0, \quad P \cdot \mathbf{v} = 0, \quad \mathbf{u}^{-1} = -\mathbf{u}, \quad \mathbf{v}^{-1} = -\mathbf{v} \quad \Longrightarrow$ $\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) \mathbf{v} \oplus \mathbf{u} \quad = \quad \mathbf{u} + \mathbf{v} \quad + \quad \frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{u}} + 1} \mathbf{u}^{-1} \cdot \frac{(\mathbf{v} \wedge \mathbf{u}^{-1})}{c^2}$

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