Categorical relativity versus relativity with Lorentz isometry group *

Zbigniew Oziewicz†
Universidad Nacional Autónoma de México
Facultad de Estudios Superiores Cuautitlán
Apartado Postal # 25, C.P. 54714 Cuautitlán Izcalli
Estado de México, México
oziewicz@servidor.unam.mx
Telephone number at home 00 52 55 2611 0352
University office telephone 00 52 55 5623 1880
University fax number 00 52 55 5870 5664

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Abstract

The categorical relativity is a groupoid category of massive bodies in mutual motions. The relative velocity is defined to be the basis-free and coordinate-free binary morphism. We are showing that coordinate-free unique definition of relative velocity in Galilean relativity becomes two different coordinate-free possibilities for relative simultaneity: binary velocity-morphism in categorical relativity, and ternary reciprocal-velocity in isometric special relativity. We are proving that the isometric Lorentz transformation needs at least three-body system. Observer-dependence and the Lorentz-covariance are different concepts.

The Poincaré-Lorentz versus the Einstein-Minkowski interpretations of a formal structure of relativity are not the unique dichotomy. We propose to consider the concept of relative velocity as the primary concept with two possibilities: Voigt 1887 & Heaviside 1888, versus Einstein 1905.

In categorical relativity the *inverse* relative-velocity-morphism $\mathbf{v}^{-1}$ is interior-observer-dependent, and not absolute as in the isometric exterior formulation where $\mathbf{v}^{-1} \equiv -\mathbf{v}$.

In the framework of categorical relativity we consider coordinate-free transformations of adopted mathematical co-frames, and transformation of proper-times (clocks), including the transformation of the Einstein-Minkowski simultaneity. The categorical relativity does not predicts the lenght/rod material contraction, because this concept is not basis-free. The concept of simultaneity is basis-free and coordinate-free, and simultaneity in categorical relativity must be relative exactly in the same way as in the isometric special relativity.

As another example we consider the electric field registered by a moving observer: electrodynamics of moving bodies is different from the isometric special relativity with Lorentz transformations.

The kinematics of categorical relativity is ruled by Frobenius algebra, whereas the dynamics of categorical relativity needs the Frölicher-Richardson algebra.

This work also review the mathematical and theoretical aspects of biological time-dilation and material length-contraction, with comparison with Langevin's interpretation in 1911.

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1 Galileo versus Descartes

René Descartes, around 1650, introduced aether conceiving space as an *absolute substance,* substratum, medium, that exists independently of any matter in it. Like a hall with sitting chairs waiting for spectators. Galileo claim instead, in 1632, that space is *relative*. The relativity theory is about the relativity of space (a set of locations), and about the relativity of time (a set of instants), and not about the change of numerical coordinates.

Why does a space, a set of locations, not need be neither absolute nor primitive concept, as it was postulated by Descartes, and also by Newton in 1686? How could it be *relative*, *i.e.* dependent on another primitive concept? What it is this primitive concept that space and time are derived? Galileo conceived space to be on massive bodies-dependent. Galilean space has no reality without the bodies that 'it contains'. Galilean primitive concept from which space is derived is the relative *motion*, the relative *velocity* among massive bodies. Because of relative velocities there are the multitude of relative spaces. If all massive bodies would be at rest relative to each other, zero relative velocities only, just one massive body, then space would be absolute and time would also be absolute. Why is space relative? Because there are relative velocities, and hence the systems of more than just one massive body in mutual motions [Galileo 1632].

In this way we come to a vicious circle: in order to define a velocity it is said, according to Newton, that we need firstly a primitive space and a primitive time, and a velocity is a derivative of a space with respect to a time. However, the very concept of a space (and a time), according to Galileo, is too much relative because of multitude of a priori relative velocities. Therefore one can ask what was first: chicken or egg? space & time or relative velocities among massive bodies? Émil Picard conceived motion and time to be dual: 'Time is measuring a motion, and a motion is measuring a time'.

Every velocity is relative, it is a velocity of one massive body relative to another massive body. This is a binary function of two bodies, like a set-valued Hom functor in category theory.

The set of all relative velocities among massive bodies is *not* a vector space of linear algebra because not every pair of such relative velocities can be composed (velocity addition is the partial operation), nor is the commutativity of composition (an additive Abelian group structure) applicable when it is defined. This holds equally well in Newtonian physics with absolute simultaneity (identified with the physical time), and as well as for relative simultaneity and finite light velocity. Moreover, every massive body possess its own identity velocity-morphism, it is the zero velocity of this object relative to himself. There is no universal unique zero velocity that would be massive-body-independent. The zero velocity of the Earth relative to Earth must *not* be identified with the zero velocity of the Sun relative to the Sun.

2 What is categorical relativity?

If relative velocity is not a vector of linear algebra, then one can ask with what mathematical concept the physical relative velocity could be identified?

2.1 Definition (Groupoid category). A category is said to be a *groupoid category*, if and only if every morphism has a two-sided inverse. In particular a *group* is a groupoid one-object-category, with just one object, hence with universal unique neutral element-morphism. A groupoid category is said to be *connected* if there is an arrow joining any two of its objects.

Each groupoid category is a quiver with inverses and with path algebra.

2.2 Definition (Terminal and initial object). An object q is terminal if to each object p there is exactly one arrow $p \to q$. An object p is initial if to each object q there is exactly one arrow $p \to q$. A *null* object is an object which is both initial and terminal.

If p is terminal, the only arrow $p \to p$ is the identity. Any two terminal objects are isomorphic [Mac Lane 1998, p. 20, p. 194].

2.3 Axiom (1: Categorical relativity is groupoid category). The categorical relativity is a connected groupoid category where *every* object is null. Each object is interpreted as some massive body, not necessarily inertial,

$$\varpi = \{\operatorname{obj}\varpi, \operatorname{arrows}\varpi\} = \{\text{massive bodies}, \text{relative velocities}\}. \tag{2.1}$$

Unique arrow from an object p to an object q is denoted by $\varpi(p, q)$, with analogy to Hom-set bifunctor. For each object $p \in \operatorname{obj}\varpi$, a map

$$\operatorname{obj}\varpi \ni q \quad \xrightarrow{\varpi_p} \quad \varpi(p, q) \in \operatorname{arrows}\varpi, \tag{2.2}$$

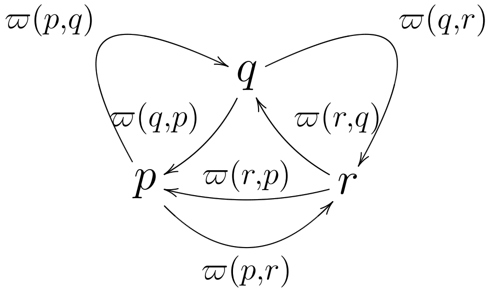
extends to covariant representable functor among connected groupoid categories. The object p is representing object for a functor ϖ_p. We call this structure the connected groupoid (1,1)-category, or enriched groupoid category.

Each relative velocity is a categorical morphism (arrow that need not be a map). An arrow $\varpi(p, q)$ is interpreted as a velocity-morphism of a body $q \in \operatorname{obj}\varpi$ relative to a body $p \in \operatorname{obj}\varpi$, *i.e.* a velocity as measured by p. We say that the source (or domain) of velocity $\varpi(p, q)$ is p, and the target (or codomain) of $\varpi(p, q)$ is a body q. A category symbol ϖ, is interpreted as an actual velocity-measuring device. We display $\varpi(p, q)$ as a categorical arrow (morphism, directed-path) which originates (is outgoing) at observer p, p is a node of the directed graph, and terminates (is incoming) at an observed body q,

$$\cdots \longrightarrow \quad p \quad \underset{\varpi(q,p)}{\overset{\varpi(p,q)}{\rightleftarrows}} \quad q \quad \longrightarrow \cdots; \tag{2.3}$$

observer of $\varpi(p, q)$ is p, observed body with $\varpi(p, q)$ is q,
observed body with $\varpi(q, p)$ is p, observer of $\varpi(q, p)$ is q.

Figure 1: A groupoid category ϖ of three massive bodies, p, q, r, with six binary velocities-morphisms. This category generate 9-dimensional algebra (Axiom 2). The identity arrows are suppressed.



Categorical relativity is (arrows ϖ)-enriched groupoid category, a connected groupoid (1,1)-category, rather then more restrictive concept of 2-category. This category is neither abelian, nor additive.

There is not yet, neither the concept of a spacetime, nor the concept of a time, nor space. The relative velocity-morphism, and massive indivisible objects, are primary, postulated primitive concepts.

Any massive body (observer, observed), is a null object in a category ϖ. The categorical null object is indivisible, like the Leibniz monad, it is not a set.

The relative *velocity* is the *primitive* notion, and this notion we are introducing as the *morphism* in the groupoid category of abstract observers. Each morphism of ϖ is a binary (interior) relative velocity. The velocity-morphism instead of the isometric Lorentz transformations in the Einstein special relativity. Why the name *categorical relativity*? Because the concept of the relative velocity we wish not associate neither with the concept of the vector space, nor with the Lorentz boost. We wish to see the relative velocity exclusively as the categorical morphism in groupoid category that is not abelian. This is why the relativity theory of space and time, in terms of the relative velocities-morphisms, is said to be the *categorical* relativity.

2.4 Axiom (2: Categorical relativity is an algebra). Let $\mathcal{F}$ be an associative, unital and commutative ring. We denote by $\operatorname{span}_{\mathcal{F}} \varpi$ the $\mathcal{F}$-module with a category ϖ as a set of basic vectors. This $\mathcal{F}$-module $\operatorname{span}_{\mathcal{F}} \varpi$ consists of all formal $\mathcal{F}$-linear combinations of the elements of ϖ, *i.e.* the formal combinations that mix objects with arrows.

It is postulated that an $\mathcal{F}$-module $\operatorname{span}_{\mathcal{F}} \varpi$ is an associative $\mathcal{F}$-algebra, called an algebra 'of massive bodies/observers in the mutual relative motions', and denoted by $\mathrm{Obs}(\varpi)$. The algebra $\mathrm{Obs}(\varpi)$ is generated by (presented on) objects and arrows of ϖ, subject to the relations: It is postulated that every object $p \in \mathrm{obj}\,\varpi$ is an idempotent $p^2 = p \in \mathrm{Obs}(\varpi)$, and every arrow of ϖ is nilpotent $(\varpi(p,q))^2 = 0 \in \mathrm{Obs}(\varpi)$. Every object of ϖ, seen in an algebras, $\mathrm{Obs}(\varpi)$ and represented as an operator in $\mathrm{End}\,(\mathrm{der}\,\mathcal{F})$, looks like a pure state in quantum mechanics.

The above postulate have the fallowing motivation. In order to have just one space, and one time, we need to fix one massive body as the reference system. The correspondence, massive body $\leftrightarrow$ idempotent, is motivated by a desire that every massive body $p \in \mathrm{End}\,(\mathrm{der}\,\mathcal{F})$, splits

$$\mathrm{der}\,\mathcal{F} = (\ker p) \oplus (\mathrm{im}\, p) = (\text{space}) \oplus (\text{time}). \tag{2.4}$$

The choice of one body, for example the Earth, as the reference system, *not* need coordinates. Such choice is coordinate-independent, and basis-independent. We call any massive body, the Earth, the Moon, an observer (no measuring devices, rods and clocks are involved).

An $\mathcal{F}$-dual $\mathcal{F}$-module is denoted by $(\mathrm{span}_{\mathcal{F}}\,\varpi)^*$. There is the distinguished covector $\mathrm{tr} \in (\mathrm{span}\,\varpi)^*$, $\mathrm{tr} : \mathrm{span}\,\varpi \to \mathcal{F}$,

$$\begin{aligned} \text{If } p \text{ is an object of } \varpi, \text{ then:} \qquad & \mathrm{tr}\, p = 1. \\ \text{If } \varpi(p,q) \text{ is an arrow of } \varpi, \text{ then:} \qquad & \mathrm{tr}\ \varpi(p,q) = 0. \end{aligned} \tag{2.5}$$

Therefore the covector tr is distinguishing an object p, from the identity arrow $\varpi(p,p)$.

For every non empty word (string) of objects, $p, q, r, \ldots, s \in \mathrm{obj}\,\varpi$, the following properties and relation are postulated,

$$1 \leq \{\mathrm{tr}(pqr \ldots s)\}^2 = \mathrm{tr}(pq)\,\mathrm{tr}(qr)\,\mathrm{tr}(r..) \ldots \mathrm{tr}(..s)\,\mathrm{tr}(sp), \tag{2.6}$$

$$1 \leq \mathrm{tr}(pq) = \mathrm{tr}(qp), \quad \mathrm{tr}(pqr) = \mathrm{tr}(qpr), \tag{2.7}$$

$$\forall\, A \in \mathrm{Obs}(\varpi), \quad qAp = \mathrm{tr}(qAp) \left\{ p + \frac{1}{c} \varpi(p,q) \right\}, \quad c < \infty. \tag{2.8}$$

From relations (2.7)-(2.8), one can deduce the algebra multiplication table, the multiplication of arrows $\{\varpi(p,q)\}$ with objects $\{p, q, r, s, ...\}$, and arrows with arrows (we set $c = 1$),

$$qp = \mathrm{tr}(qp)\{p + \varpi(p,q)\}, \tag{2.9}$$

$$q\varpi(p,r) = \left(\frac{\mathrm{tr}(qrp)}{\mathrm{tr}(rp)} - \mathrm{tr}(pq)\right)\{p + \varpi(p,q)\}, \tag{2.10}$$

$$\varpi(q,r)p = \frac{\mathrm{tr}(rqp)}{\mathrm{tr}(rq)}\{p + \varpi(p,r)\} - \mathrm{tr}(qp)\{p + \varpi(p,q)\}, \tag{2.11}$$

$$\begin{aligned}\varpi(p,q)\varpi(r,s) = \frac{1}{\mathrm{tr}(qp)}&\left(\frac{\mathrm{tr}(qpsr)}{\mathrm{tr}(sr)} - \mathrm{tr}(qpr)\right)\{r + \varpi(r,q)\}\\ &+\left(\mathrm{tr}(pr) - \frac{\mathrm{tr}(psr)}{\mathrm{tr}(sr)}\right)\{r + \varpi(r,p)\}.\end{aligned} \tag{2.12}$$

The relative velocity is universal velocity-valued function of two variable-bodies. In categorical relativity there is no need to distinguish separately the constant velocities (special relativity) from the variable accelerated velocities, hence the categorical relativity goes beyond boundary of the special relativity.

Zero velocity of a body p relative to himself is $\varpi(p,p) = \mathbf{0}_p$. All these zero velocities are equal to unique zero of algebra $\mathrm{Obs}(\varpi)$ for all bodies, however they can *not* be identified with respect to associative composition of morphisms in a category ϖ, see below.

In [Cruz & Oziewicz 2006] we consider augmented unital algebra $(\mathrm{Obs}\,\varpi)\oplus\mathcal{F}$, and pose a hypothesis that this trace-class algebra is a Frobenius algebra for each cardinality of family of objects.

2.5 Axiom (3: Scalar magnitude of arrow). The arrows of the category ϖ does not possess a linear structure (an arrow multiplied by a scalar is not an arrow), therefore the Euclidean structure of the linear algebra, the tensor product, is meaningless for them. The linear structure of an $\mathcal{F}$-algebra $\mathrm{Obs}\,\varpi$ is useless for the definition of the scalar magnitude of an arrow (and for defining an angle among arrows) because every arrow seen in an algebra $\mathrm{Obs}\,\varpi$ is nilpotent. An Euclidean angle between arrows with the same source only, and a scalar magnitude of each arrow, needs the following separate postulate

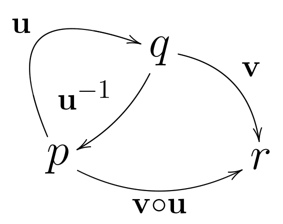
$$\frac{\varpi(q,p)\cdot\varpi(q,r)}{c^2} \equiv 1 - \frac{\mathrm{tr}(pr)}{\mathrm{tr}(pqr)} \quad\Rightarrow\quad \left(\frac{|\varpi(p,q)|}{c}\right)^2 = 1 - \frac{1}{\mathrm{tr}(pq)}. \tag{2.13}$$

The above definition (2.13) is meaningful for $c < \infty$ only.

2.6 Theorem (Associative addition of relative velocities). *See* Figure 2 . *Set* $\mathbf{u} \equiv \varpi(p,q)$, $\mathrm{source}(\mathbf{u}) = p$, *and* $\mathbf{v} \equiv \varpi(q,r)$. *We adhere to the arabic convention of the composition of morphisms, read from the right to the left,*

$$\varpi(q,r)\circ\varpi(p,q) = \varpi(p,r) = \mathbf{v}\circ\mathbf{u}. \tag{2.14}$$

Figure 2: Associative composition of velocities-morphisms.



The binary composition of composable arrows is associative and has the following p-dependent explicit form,

$$(1 - \tfrac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2})(\mathbf{v} \circ \mathbf{u}) = \mathbf{u} + (1 - \tfrac{\mathbf{u}^2}{c^2})\mathbf{v}p + \tfrac{1}{c}(\mathbf{v} \cdot \mathbf{u}^{-1})p. \tag{2.15}$$

Proof. The velocity-addition follows from the associativity of an algebra Obs ϖ, $r(qp) = (rq)p$, and from Axiom 2, formula (2.8),

$$\begin{aligned} r(qp) &= \operatorname{tr}(qp)\operatorname{tr}(rp)\{\varpi(p,r) + p\} + \operatorname{tr}(qp)r\varpi(p,q),\\ (rq)p &= \operatorname{tr}(qp)\operatorname{tr}(rq)\{\varpi(p,q) + r\} + \operatorname{tr}(rq)\varpi(q,r)p. \end{aligned} \tag{2.16}$$

Therefore

$$\begin{aligned} \operatorname{tr}(qp)\operatorname{tr}(rp)\varpi(p,r) = \operatorname{tr}(qp)\operatorname{tr}(rq)\varpi(p,q) + \operatorname{tr}(r,q)\varpi(q,r)p \\ + \operatorname{tr}(qp)\operatorname{tr}(r(q-p))p - \operatorname{tr}(qp)r\varpi(p,q). \end{aligned} \tag{2.17}$$

The multiplication of an object r with an arrow $\varpi(p,q)$, as given by formula (2.10), is expressed in terms of $\{\varpi(p,r) + p\}$. This gives

$$\begin{aligned} \varpi(p,r) &= \varpi(q,r) \circ \varpi(p,q) \\ &= \tfrac{\operatorname{tr}(pqr)}{\operatorname{tr}(pr)}\left(\varpi(p,q) + \tfrac{1}{\operatorname{tr}(pq)}\varpi(q,r)p\right) + c\left(\tfrac{\operatorname{tr}(pqr)}{\operatorname{tr}(pr)} - 1\right)p. \end{aligned} \tag{2.18}$$

Finally we need use Axiom 3, expression (2.13). In [Oziewicz 2005] there is an equivalent form of this associative velocity addition. □

The Einstein addition of velocities in isometric special relativity is non-associative, as observed by Ungar [Ungar 1988, 2001]. In contrast, the addition of velocities in categorical relativity, Theorem 2.6 and expression (2.15), is observer-dependent (depends on observer p that is the source of the velocity-arrow $\mathbf{u}$), however this addition operation is associative.

If $\varpi(p,q)$ is a velocity of q relative to p, then the $\circ$-inverse velocity-morphism, $\varpi(q,p)=(\varpi(p,q))^{-1}$, is a velocity of p relative to q.

$$\begin{aligned}\varpi(p,q)\circ\varpi(q,p)=\mathbf{0}_q \quad \neq \quad & \mathbf{0}_p=\varpi(p,p)=\varpi(q,p)\circ\varpi(p,q)\\ &=\operatorname{tr}(pq)\varpi(p,q)+\varpi(q,p)p+c(\operatorname{tr}(pq)-1)p. \quad (2.19)\end{aligned}$$

2.7 Example (Galilean addition). Two objects, p and q, possess the same simultaneity iff $\operatorname{tr}(pq)=1$. Therefore in the Galilean algebra of observers the trace of arbitrary string of objects must be $\operatorname{tr}(pq\ldots)=1$. The Galilean algebra of observers is presented on idempotent-objects, and on the following relations, not independent,

$$\begin{aligned}qp=q=p+\varpi(p,q), &\qquad \varpi(p,q)\varpi(r,s)=0,\\ q\varpi(p,r)=0, &\qquad \varpi(q,r)p=\varpi(q,r).\end{aligned} \quad (2.20)$$

Hence Axiom 2 with (2.8) must be replaced by the reciprocal relative velocity $\varpi(p,q)=q-p$, *i.e.* relative velocity in Galilean relativity is exactly the *skew* symmetric function of his arguments,

$$\varpi(q,r)\circ\varpi(p,q)=\varpi(p,q)+\varpi(q,r)=(q-p)+(r-q)=\varpi(p,r). \quad (2.21)$$

Therefore, $\lim_{c\to\infty}(\mathbf{v}\circ\mathbf{u})=\mathbf{v}+\mathbf{u}$, because $\mathbf{v}p=\mathbf{v}$.

3 Module over an algebra of observers

An $\mathcal{F}$-algebra of observers, $\operatorname{Obs}(\varpi)$, is abstractly isolated from the concepts of spacetime, time and space. The indivisible objects of ϖ are not yet located neither in a space, nor in a time. A ring $\mathcal{F}$ is interpreted as an $\mathbb{R}$-algebra of the classical measurements, in case that $\mathcal{F}$ is commutative.

The spacetime manifold of events is usually identified with $\operatorname{alg}(\mathcal{F},\mathbb{R})$. However we prefer to see the concept of the spacetime encoded in an $\operatorname{Obs}(\varpi)$-module, *i.e.* in an algebra morphism, from an associative $\mathcal{F}$-algebra $\operatorname{Obs}\varpi$, into endomorphism algebra of the Lie $\mathcal{F}$-module $\operatorname{der}\mathcal{F}$,

$$\operatorname{Obs}(\varpi) \quad \xrightarrow[\text{morphism}]{\text{associative algebra}} \quad \operatorname{End}(\operatorname{der}\mathcal{F})\simeq(\operatorname{der}\mathcal{F})\otimes(\operatorname{der}\mathcal{F})^*. \quad (3.1)$$

The Lie $\mathcal{F}$-module $\operatorname{der}\mathcal{F}$ of derivations of the ring $\mathcal{F}$, and the dual $\mathcal{F}$-module, $(\operatorname{der}\mathcal{F})^*$, of the differential forms, are considered as $\operatorname{Obs}(\varpi)$-modules.

A massive body $p\in\operatorname{obj}\varpi$ is represented by $(1,1^*)$-tensor field $p\in\operatorname{End}(\operatorname{der}\mathcal{F})$, where $p^2=p$ must be the minimal polynomial. A pull back (transpose) of p is denoted by p^*, it is a $(1^*,1)$-tensor field.

Every object of ϖ, seen in an algebras, $\mathrm{Obs}(\varpi)$ and $\mathrm{End}\,(\mathrm{der}\,\mathcal{F})$, looks like a pure state in quantum mechanics. In order to have just one space, and one time, we need to fix one massive body as the reference system. The correspondence, massive body $\leftrightarrow$ idempotent, is motivated by a desire that every massive body $p \in \mathrm{End}\,(\mathrm{der}\,\mathcal{F})$, splits

$$\mathrm{der}\,\mathcal{F} = (\ker p) \oplus (\mathrm{im}\,p) = (\mathrm{space}) \oplus (\mathrm{time}). \tag{3.2}$$

The choice of one body, for example the Earth, as the reference system, *not* need coordinates. Such choice is coordinate-independent, and basis-independent. We call any massive body, the Earth, the Moon, an observer (no measuring devices, rods and clocks are involved).

The gravity potential tensor field $g = g^* \in (\mathrm{der}\,\mathcal{F}) \otimes (\mathrm{der}\,\mathcal{F})$, and his inverse $g^{-1} \in (\mathrm{der}\,\mathcal{F})^* \otimes (\mathrm{der}\,\mathcal{F})^*$, are considered as an invertible Grassmann $\mathcal{F}$-algebra morphisms from differential multi-forms to multi-vector fields $(\mathrm{der}\,\mathcal{F})^\wedge$,

$$n^* \xrightarrow{\;p^*\;} n^* \underset{g^{-1}}{\overset{g}{\rightleftarrows}} n \xrightarrow{\;p\;} n, \qquad \forall\, n \in \mathbb{N}. \tag{3.3}$$

In this convention, $1 \in \mathbb{N}$ denotes (a grade of) a Lie $\mathcal{F}$-module of vector fields $\mathrm{der}\,\mathcal{F}$, $2 \in \mathbb{N}$ denotes an $\mathcal{F}$-module of bivector fields $(\mathrm{der}\,\mathcal{F}) \wedge (\mathrm{der}\,\mathcal{F})$, 1^* is a module of Pfaffian differential one-forms $(\mathrm{der}\,\mathcal{F})^*$, etc. Therefore $g \in \mathrm{alg}(n^*, n)$, however, $\mathrm{obj}(\varpi) \ni p \leadsto p \in \mathrm{der}(n,n)$. Here *leadsto* means 'extends', and $\mathrm{der}(n,n)$ is short for $\mathrm{der}((\mathrm{der}\,\mathcal{F})^{\wedge n}, (\mathrm{der}\,\mathcal{F})^{\wedge n})$.

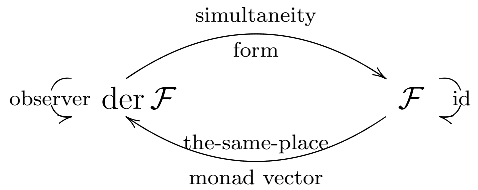
The composition, $p \circ g$, and their transpose, $(p \circ g)^* = g \circ p^*$, are both morphism with the same domain and codomain, and therefore one can ask that a massive body $p \in \mathrm{End}\,(\mathrm{der}\,\mathcal{F})$ is metric-compatible (g-orthogonal),

$$p^2 = p \quad \Longrightarrow \quad \mathrm{tr}\,p = \dim \mathrm{im}\,p,$$
$$p^2 = p \quad \& \quad \mathrm{tr}\,p = 1 \quad \& \quad g \circ p^* = p \circ g \quad \Longleftrightarrow \quad p = \frac{P \otimes (g^{-1}P)}{g^{-1}(P \otimes P)}. \tag{3.4}$$

The above compatibility of observer p with the gravitational potential g (3.4), is postulated in the present paper, however we believe that this compatibility must be tested experimentally and not postulated a priori.

Let $g = g^*$ be a Lorentzian metric tensor field of signature $(-+++)$. We do not suppose that g must be necessarily curvature-free. An idempotent is said to be time-like if his every non-zero eigenvector (a monad field) is time-like, $pP = P \;\Rightarrow\; P^2 < 0$. The space-like simultaneity of an observer p must be given by the Einstein proper time, $-g^{-1}P$. This time-like differential Pfaff

Figure 3: An observer as the split-idempotent.



$$p = (\text{monad vector}) \otimes (\text{simultaneity form})$$

form, $-g^{-1}P$, encode the unique empirical and metric-dependent simultaneity of an observer p. Metric-compatible observer is conformally invariant.

In what follows we set all monad fields (the eigenvectors of observers) to be normalized, $pP = P$ with $P^2 = -1$.

The notion of observer is similar as e.g. in Gottlieb [1996]. For more motivations we refer also to [Cruz and Oziewicz 2003]. The reference-frame as used by Llosa and Soler [2004], need space metric, and therefore is not a basis, it is not a frame.

4 Binary relative velocity

Consider two-body massive system,

$$p \equiv P \otimes (-g^{-1}P) \quad \text{and} \quad q \equiv Q \otimes (-g^{-1}Q), \tag{4.1}$$
$$P^2 = Q^2 = -1, \quad \mathrm{tr}(pq) = (P \cdot Q)^2. \tag{4.2}$$

4.1 Corollary (Binary relative velocity). An arrow of a category ϖ, a velocity-morphism from an observer p to observed q, is represented in an algebra End (der $\mathcal{F}$) as the space-like velocity field of q relative to p,

$$\varpi(p,q) = \varpi_g(P,Q) \otimes (-gP), \tag{4.3}$$
$$\frac{1}{c}\varpi_g(P,Q) \equiv \frac{(\mathrm{id} - p)Q}{(-g^{-1}P)Q} = \frac{Q}{-P \cdot Q} - P \quad \in \mathrm{der}\,\mathcal{F}, \tag{4.4}$$
$$\left(\frac{\varpi(P,Q)}{c}\right)^2 = 1 - \frac{1}{(P \cdot Q)^2}. \tag{4.5}$$

4.2 Corollary (Light velocity). The light propagation is given by the light-like vector field, $L^2 = 0$. The magnitude of the speed of the light is observer-independent

$$\mathbf{c} \equiv \varpi_g(P, L) = \left(\frac{L}{-P \cdot L} - P\right) c \quad \Longrightarrow \quad \mathbf{c}^2 = c^2. \tag{4.6}$$

The most of textbooks teach that 'all inertial observers measure the same speed of light', cf. also with [Dvoeglazov & Quintanar González 2006, Conclusions]. Such statement could suggest that non-inertial observers would measure observer-dependent speed of light. Our Corollary 4.2 is stronger: categorical relativity predict that speed of light is independent of *arbitrary* observer, including *all* non-inertial, rotating and accelerating observers.

4.3 Corollary (Heaviside-FitzGerald-Lorentz scalar factor). Set $\mathbf{v} \equiv \varpi_g(P, Q)$ and $\mathbf{v}^{-1} \equiv \varpi_g(Q, P)$. An assumption that an observer p is g-orthogonal, imply that $P \cdot \mathbf{v} = 0$,

$$P \cdot \mathbf{v} = 0, \quad Q \cdot \mathbf{v}^{-1} = 0, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{\mathbf{c}^2}}} = -P \cdot Q, \tag{4.7}$$

$$Q = \gamma(\tfrac{\mathbf{v}}{c} + P) \quad \text{and} \quad P = \gamma(\tfrac{\mathbf{v}^{-1}}{c} + Q), \tag{4.8}$$

$$\tfrac{\mathbf{v}}{c} = \tfrac{Q}{\gamma} - P = -\gamma \tfrac{\mathbf{v}^{-1}}{c} - (\gamma - \tfrac{1}{\gamma})Q = -\tfrac{\mathbf{v}^{-1}}{\gamma c} - (1 - \tfrac{1}{\gamma^2})P, \tag{4.9}$$

$$\tfrac{\mathbf{v}^{-1}}{c} = \tfrac{P}{\gamma} - Q = -\gamma \tfrac{\mathbf{v}}{c} - (\gamma - \tfrac{1}{\gamma})P = -\tfrac{\mathbf{v}}{\gamma c} - (1 - \tfrac{1}{\gamma^2})Q. \tag{4.10}$$

4.4 Corollary (Categorical boost). Denote $\varpi(p, q) = \mathbf{v} \otimes (-g^{-1}P)$ by ϖ. The coordinate-free and basis-free transformation among bodies-idempotents in relative motion, the categorical boost, is as follows, $c < \infty$,

$$p \underset{\varpi(q,p)}{\overset{\varpi(p,q)}{\rightleftarrows}} q = q(p, \mathbf{v}) =$$

$$= \gamma^2 \left\{ p + \frac{1}{c}(\varpi + g \circ \varpi^* \circ g^{-1}) + \frac{1}{c^2} \varpi \circ g \circ \varpi^* \circ g^{-1} \right\} \tag{4.11}$$

$$= \gamma^2 \left\{ p - \frac{1}{c}(P \otimes g^{-1}\mathbf{v} + \mathbf{v} \otimes g^{-1}P) - \frac{1}{c^2} \mathbf{v} \otimes g^{-1}\mathbf{v} \right\}, \tag{4.12}$$

$$q^* \circ p^* = -\gamma\, (g^{-1}Q) \otimes P, \qquad q^* \circ (1^* - p^*) = -\gamma\, (g^{-1}Q) \otimes \mathbf{v}. \tag{4.13}$$

Proof. Insert (4.8) into (4.1). □

4.5 Theorem (Binary velocity is not reciprocal). *The binary velocity* (4.4) *can not be reciprocal because simultaneity is not absolute,*

$$|\mathbf{v}| = |\mathbf{v}^{-1}|, \qquad \left(\mathbf{v}^{-1} + \mathbf{v}\right)^2 = -2\frac{\gamma^2 - 1}{\gamma} c^2,$$

$$\left(\mathbf{v}^{-1} \wedge \mathbf{v}\right)^2 = -\frac{(\gamma^2 - 1)^3}{\gamma^4} c^4. \tag{4.14}$$

Definition 4.4 of the relative velocity is implicit in [Minkowski 1908], and in adopted system of coordinates in [Vargas 1982, p. 770, formula (29)]. This definition as a coordinate-free and basis-free projection we proposed in [Świerk 1988]. The same definition was introduced independently by Matolcsi [1993, p. 191], Matolcsi and Goher [2001, p. 89, Definition (18)], Gottlieb [1996], Mitskievich [2005, formula (4.19) on p. 16].

5 Lorentz group versus groupoid category

At this stage reader could ask how groupoid category is related with concept of a group? How it is related with the Lorentz and Poincaré groups of symmetries of the flat metric of the empty space-time, of the Einstein's special relativity [Einstein 1905]? In a group and in a groupoid category, there is *always* the unique inverse operation. However, there are two differences among the concept of a group, and a concept of a groupoid category.

- The group binary associative operation is *global*. In the group every pair of elements can be composed. The morphisms in a category, if they can be composed, are composed associatively. However, in a category every morphism has the definite source object and the definite target object. Every physical relative velocity is a velocity of the concrete target body relative to another source body. Not all velocities-morphism are composable. The composition of categorical morphisms is the *partial* binary associative operation. For example, in categorical relativity, we are *not* allowed to add the velocity of the Sun relative to Earth, with the velocity of the Moon relative to the Earth. These velocities-morphisms are not composable arrows. However such addition is allowed in Einstein' special relativity where one can compose within the Lorentz group every Lorentz boost with another arbitrary Lorentz boost.

- Group possess the unique neutral element. Contrary to this in category every object (every massive body) posses his own zero velocity-morphism.

In order to pass from categorical relativity of the groupoid category of massive objects, to Einstein's special relativity, it is necessary (but not sufficient) to *choose one massive* object ($e =$ Earth?) to be preferred object. The Cosmic Microwave Background radiation is massless and therefore can not be considered as the preferred observer. There is no relativity theory for massless light-like radiation. Massless radiation does not possess a space-time split. The choice of a preferred massive body, $e \in \text{obj}\,\varpi$, will pick up the unique zero velocity $\mathbf{0}_e$ (neutral element of a group). In this case all binary velocities $\{\varpi(p,q)\}$ can be 'projected on' (measured by) the preferred body, and this leads to the several distinct concepts of ternary relative velocities, $\varpi(e,p,q) \in \ker e$.

5.1 Definition (Ternary relative velocity). Let q, p, e be a system of three massive bodies. The velocity of a body q relative to p, as measured/seen by an external (preferred) body e, $\varpi(e,p,q) \in \ker e$, is said to be the ternary relative velocity. Ternary velocity is said to be *reciprocal* if

$$\varpi(e,p,q) = -\varpi(e,q,p). \tag{5.1}$$

Ternary velocity is said to be isometric-image if

$$|\varpi(e,p,q)| = |\varpi(q,p)|. \tag{5.2}$$

In contrast to the unique binary relative velocity that is always the traceless arrow of ϖ, and is never reciprocal, there are several distinct concepts of a ternary velocities, depending whether it is, or it is not, reciprocal, traceless, isometric-image, an arrow of ϖ.

5.2 Theorem (Lorentz-boost-link problem). *The* reciprocal *ternary relative velocity is equivalent to the isometric Lorentz boost. Einstein's isometric exterior formulation needs ternary velocities that in general are not isometric-image, and possess non-associative addition.*

Proof. The proof was announced on several meetings [Oziewicz 2005]. The clue for understanding are: the reciprocity of the inverse velocity (*assumed* by Einstein in 1905), and the Lorentz-boost-link problem raised by van Wyk in 1986. □

5.3 Clarification (History of the concept of velocity). Since Galileo the velocity was defined in terms of the *measurement* as the ratio of the distance to time, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$. Such coordinate definition need the primary concepts of a space, time, and the concept of a curve in the space [Lévy-Leblond 1980]. The most essential property of the physical velocity, its relativity, is lost, or at least it is not explicit when using coordinates. Coordinates do not need massive bodies. The velocity of the car relative to the street, $\varpi(\text{street}, \text{car})$, is not the same as the velocity of the car relative to the car, $\varpi(\text{car}, \text{car}) = \mathbf{0}_{\text{car}}$.

According to Einstein [1905] and Minkowski [1909], the velocity of the car relative to the street is given by Lorentz isometric transformation,

$$L(\mathbf{v})(\text{street}) = (\text{car}), \qquad \text{see Section 7.} \tag{5.3}$$

Van Wyk in 1986 try to solve the Lorentz-boost-link equation (5.3) for unknown velocity $\mathbf{v}$, for the given 'car' and the given 'street', and found that the solution is not unique! We showed elsewhere that in order to fix one solution for the Lorentz-boost-link one needs third massive body, an exterior observer. That exterior massive observer is needed it is clear also because the Lorentz transformation imply that the velocity is reciprocal, *i.e.* the velocity of the street relative to car must be $-\mathbf{v}$, $L(-\mathbf{v}) \circ L(\mathbf{v}) = \text{id}$. Therefore the Einstein relative velocity is ternary, $\mathbf{v} = \mathbf{v}(\text{exterior observer}, \text{street}, \text{car})$.

Gill and Zachary since 1987, and Gill, Zachary, Lindesay 1997-2001, proposed to define a space-like velocity in terms of metric-*dependent* proper-time, *i.e.* the Minkowski proper-time restricted to the given curve. Such definition does not need the coordinate system on four-dimensional space-time, but needs metric. Still the property of the velocity to be relative is lost or it is hidden.

The coordinate expressions for the introduced in the present paper the binary and ternary velocities will be presented elsewhere.

In categorical relativity two physical massive bodies are considered to be different objects if and only if there is no zero primitive relative velocity between them. The physical 'two-body system' with zero relative velocity among these bodies, is considered to be one object. The two-body decay and a two-body collapse, *i.e.* the groupoid category with variable number of objects, will be considered in a separate paper.

Readers believing that 'experiments have the last word' will ask: how a velocity is measured in theory where a relative velocity is a primitive theoretical concept? or: how a mass is measured in different gravity *theories*? The experimental measurements are theory-independent, are theory-*free*. These measurements, however, could be explained in many different theories, and find different interpretations within different theories. Compare with the last sentence in [Jefimenko 1998]: 'it is fallacious to interpret experiment as a proof of length contraction or time dilation as long as the experiment has a simple and clear alternative interpretation'. There is no difference among experimental measurement of the relative velocity in the Galilean theory, in the Einstein' special relativity, and in categorical relativity. Why the experimental measurement must be theory-dependent? The experiment does not have the 'last word', because the 'last word experiment' can be reinterpreted in many *different* theories many centuries after this experiment was done. There is no one-to-one map among experimental data and theories. The last word never can be said. Carl Popper in 1934 was trying distinguish 'objectively' among science and pseudo-science. According to Popper: theory is scientific if and only if there is a crucial experiment that could 'in principle' *disprove* this theory. All experiments that confirm a given theory, never can be considered as the last-word-proof of the absolute validity of the given theory. Karl Popper was rightly criticized by Imre Lakatos [1973, 1983], and by Paul Feyerabend [1975]. The aim of the present note is restricted for the brief presentation of the mathematical conceptual formalism only. Referee is right demanding that the proposed here the categorical relativity theory 'should be discussed with experimental tests'. Does this means, that the mathematical formalism must be prohibited for publication if not positively tested experimentally before? The reinterpretation of old experiments, and new test proposals, for the proposed theory is outside of the scope of the present paper.

A coordinate time transformation, *i.e.* a relativity of coordinate proper time, was considered by Potier in 1874, and by Voigt at Königsberg and Göttingen in 1887 [Voigt 1887, Ernest and Hsu 2001]. Nowadays, in XXI century, neither Potier, nor Voigt, are accepted as the forerunners of special relativity, notwithstanding that they introduced relative coordinate time, because they consider *not* reciprocal coordinate time transformations

$$\{x, t, x', t', \mathbf{v}\} \quad \not\Longleftrightarrow \quad \{x', t', x, t, -\mathbf{v}\}. \tag{5.4}$$

Relativity of simultaneity was raised by Poincaré [1904], and considered by Einstein [1905]. In categorical relativity the transformations among reference systems, Corollary 4.4, do not satisfy reciprocity (as in Potier, and in Voigt),

$$\{p, q, \mathbf{v}\} \quad \Longleftrightarrow \quad \{q, p, \mathbf{v}^{-1} \neq -\mathbf{v}\} \tag{5.5}$$

This is contrary to Einstein's special relativity with reciprocal Lorentz isometric transformations.

Albert Einstein concluded in 1905, that the morphism among reference systems in mutual motion must be necessarily the Lorentz group isometry transformation [Einstein 1905, Minkowski 1908]. Note that, it is the Lorentz group (independent of number of bodies) in special relativity, that is exchanged to groupoid category that depends on number of objects.

Assume that inertial reference systems (the massive bodies, observers) are in mutual constant relative motion. Albert Einstein started his 1905-paper with two postulates:

- The relativity principle stated by Copernicus [1543], Galileo [1632], Poincaré [1904], and Lorentz [1904]: physical laws, phenomenons, are observer-independent. Einstein added: including electromagnetic phenomena, $dF = 0$ & $dJ = 0$.

- Light velocity is observer-independent. Compare with Corollary 4.2.

Léon Brillouin [1970, page 71], share the following opinion, 'Lorentz transformations is definitely not physical'. We showed elsewhere, and in another way, that two Einstein's postulates are not equivalent to Lorentz-group-transformations among reference systems [Oziewicz 2005]. The clue for understanding this fact are: the reciprocity of the inverse velocity (*assumed* by Einstein in 1905), and the Lorentz-boost-link problem raised by van Wyk in 1986.

In categorical relativity, the Lorentz invariance/covariance is neither broken, nor violated. It is inapplicable, because the binary velocity-projection (4.4) can not parameterize the Lorentz isometric boost. In categorical relativity the Lorentz/Poincaré/Galilei group-covariance/invariance is conceptually *separated* from observer-independence.

- The main problem of Einstein's isometric special relativity: does the considered physical concept (e.g. spin-density, electromagnetic field, density of the electric charge, etc), is, or it is not, Lorentz-covariant, or Lorentz-invariant? What means Lorentz violation in field theory [Mattingly 2005]?
- The main problem of the categorical relativity: does the considered physical concept, is, or it is not, observer-free?

Categorical relativity could collapse to the Einstein special relativity when a preferred massive body is selected. The categorical relativity is an alternative mathematical theory where spacetime is defined to be an $\mathrm{Obs}(\varpi)$-module, that does not exists without massive bodies and their mutual relative velocities-morphisms. In particular there are no empty space-time. The categorical relativity is *not* against the Einstein special relativity: two distinct concepts of the relative velocity, the binary primary velocity-morphism of categorical relativity, and the ternary relative velocity of the Einstein's isometric special relativity, are related by means of the natural transformation of functors among groupoid categories.

In this note we are showing that in the categorical relativity the transformation of the electric and magnetic fields relative to moving observer, is different, in comparison with transformation first discovered by Lorentz and accepted by Einstein.

6 Electromagnetic field: Lorentz-covariant? absolute?

In what follows $F \in (\operatorname{der} \mathcal{F})^* \wedge (\operatorname{der} \mathcal{F})^*$, is a closed differential biform of the electromagnetic field, $dF = 0$. In categorical relativity the electromagnetic field F is observer-independent for every observer, including non-inertial, *i.e.* F is absolute. This observer-independence must be contrasted with the Lorentz-covariance of F in the Einstein's special relativity. We need to recall that the Lorentz group (and the inhomogeneous Poincare group), is a group of isometries of the zero-curvature metric tensor field $g \in (\operatorname{der}\mathcal{F})^{\otimes 2}$, possessing six or ten Killing vector fields, $O_g = \operatorname{Iso}(g)$. What is domain of the action of the Lorentz group? Most textbooks present the Lorentz transformations primarily as the transformations of coordinates, *i.e.* as a transformation of the set of scalar fields $\{x, t, \ldots\} \in \mathcal{F} \times \mathcal{F} \times \ldots$. Every tensor is coordinate-free, e.g. the metric tensor g, the electromagnetic tensor F, etc. Therefore arbitrary transformation of coordinates must leave every tensor invariant, and in this respect the particular Lorentz transformations of coordinates must be irrelevant. To be an isometry, the domain of the Lorentz transformation must be an $\mathcal{F}$-modul $\operatorname{der}\mathcal{F}$. The Lorentz group must act on every vector field in $\operatorname{der}\mathcal{F}$, in such way that the scalar product will be invariant,

$$O_g \equiv \operatorname{Iso}(g) \equiv \{L \in \operatorname{aut}_{\mathcal{F}}(\operatorname{der}\mathcal{F}), \forall\, X, Y \in \operatorname{der}\mathcal{F},\\ X \cdot Y \equiv g^{-1}(X \otimes Y) = (LX) \cdot (LY).\} \quad (6.1)$$

The set of all frames in $\operatorname{der}\mathcal{F}$ is denoted by $f \operatorname{der}\mathcal{F}$, f is a covariant functor. Every basis in $\operatorname{der}\mathcal{F}$, is an element of $f \operatorname{der}\mathcal{F}$. A concept of a basis is mathematical, all tensors are basis-free. When Lorentz transformation $L \in O_g$ act on every vector, this is said to be an 'active' transformation. Every action on $\operatorname{der}\mathcal{F}$ extends to an action on all frames (basses) in $\operatorname{der}\mathcal{F}$. An action of the Lorentz group on $f \operatorname{der}\mathcal{F}$ is said to be 'passive'. The names, passive and active, are misleading, because in fact the domain of the action is changed, from $\operatorname{der}\mathcal{F}$ to $f \operatorname{der}\mathcal{F}$, and every action is active on his domain. However when considering an action of Lorentz transformation on frames $f \operatorname{der}\mathcal{F}$, it is supposed that in this case group does not act on vectors in $\operatorname{der}\mathcal{F}$, and therefore every change of a frame induce a new representation of every vector in a new frame, *i.e.* the group is acting on coordinates of vectors.

One can ask where is the isometry condition (6.1) hidden in the action on frames? In order to restrict from the group of all module automorphisms $\operatorname{Aut}(\operatorname{der}\mathcal{F})$, acting on frames, to the Lorentz group O_g, one needs artificial restriction to subset of g-orthogonal Lorentz-frames. However the very concept of a frame, a basis, is a mathematical convenience, and every physically meaningful tensor, as basis-free concept, can be described in any non-orthogonal frame, without the change of the physical contents. From this point of view the 'passive' action of the Lorentz group on mathematical frames is useless for physical phenomenons. For example, Landau and Lifshitz in their monograph *The Classical Theory of Fields*, 1973 and 1975 edition, noted that 'the choice of *not* Lorentz frame could be more convenient mathematically'.

Therefore logically acceptable is to consider 'active' Lorentz transformations of $\mathcal{F}$-modul $\operatorname{der}\mathcal{F}$, only, disregarding mathematical frames [Mattingly 2005, p. 7]. In this case metric tensor field g is Lorentz-invariant, however each vector field in $\operatorname{der}\mathcal{F}$, will be transformed, it is Lorentz-'covariant'. Similarly, entire tensor algebra build from $\mathcal{F}$-module $(\operatorname{der}\mathcal{F})^*$, including electromagnetic Faraday tensor field F will be actively transformed by Lorentz group.

Therefore the observer-independence of electromagnetic field F, in categorical relativity, is conceptually different from Lorentz-covariance of F in Einstein's special relativity.

The action of an isometry L on $\mathcal{F}$-module $\operatorname{der}\mathcal{F}$, extends to the Grassmann algebra action on multivector fields, and a pull-back L^* act on Grassmann algebra of differential forms $(\operatorname{der}\mathcal{F})^{*\wedge}$. All these tensor fields are Lorentz-covariant.

7 Isometry from bivector

Let α and β be differential multiforms. Then we use the following notation for their Grassmann exterior product. The regular right representation of the Grassmann algebra in endomorphism algebra is denoted by

$$\text{Grass} \xrightarrow{\;e\;} \text{End}\,(\text{Grass}), \qquad e_\alpha \beta \equiv \alpha \wedge \beta. \tag{7.1}$$

Let $P \in \operatorname{der}\mathcal{F}$, be a vector field, then $i_P = (e_P)^* \in \operatorname{der}((\operatorname{der}\mathcal{F})^{*\wedge})$, is a graded derivation of Grassmann algebra of differential forms. Let $b \in (\operatorname{der}\mathcal{F})\wedge(\operatorname{der}\mathcal{F})$, be a simple bivector field. Then i_b is the composition of two derivations, that is not derivation. However i_b is a well defined contraction acting on differential multiforms. In what follows the following assumptions and notations will be used for a bivector b, and for a vector P,

$$b \wedge b = 0, \quad b^2 \leq 1 \quad \text{and} \quad \lambda \equiv 1 + \sqrt{1 - b^2} \;\in \mathcal{F}, \tag{7.2}$$

$$V \equiv \tfrac{1}{\lambda} i_{g^{-1}b}(P \wedge b) - i_{g^{-1}P} b - \tfrac{b^2}{\lambda} P \quad \in \operatorname{der}\mathcal{F}. \tag{7.3}$$

The Lie algebra of isometry group O_g is given by bivectors. Consider the isometry $L_b \in O_g$, generated by a simple bivector b as assumed in (7.2). Then, $(L_b)^{-1} = L_{-b}$, and we can state the technical Lemma.

7.1 Lemma. *Using abbreviation* (7.3) *for a vector field* V, *the following commutator expression holds on multiforms,*

$$i_P \circ L_b^* - L_b^* \circ i_P = i_V + \tfrac{2}{\lambda}\, e(i_P g^{-1} b) \circ i_b. \tag{7.4}$$

We will apply Lemma 7.1 for the following particular bivector, where $\mathbf{v}$ is the reciprocal *ternary* velocity parameterizing the Lorentz boost, cf with (4.7),

$$P^2 = -1, \quad P \cdot \mathbf{v} = 0, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{\mathbf{c}^2}}} \neq -P \cdot Q,$$
$$b = \gamma P \wedge \mathbf{v}, \quad b^2 = -(\gamma^2 - 1), \quad \lambda = \gamma + 1, \tag{7.5}$$
$$i_P \circ L_b^* = \gamma\, i_{P+\mathbf{v}} - \left\{ (e \circ g^{-1}) \left(P + \frac{\gamma}{\gamma + 1} \mathbf{v} \right) \right\} \circ i_b. \tag{7.6}$$

8 Electric field and magnetic field: observer-dependent? Lorentz-covariant?

In Einstein's special relativity the electric field is the strange concept: Lorentz-covariant? in what meaning? In categorical relativity the electric and magnetic fields are coordinate-free, basis free, however they are observer-dependent, *i.e.* these fields are concomitants of the observer-tensor.

The observer, $p \in \mathrm{End}\,(\mathrm{der}\,\mathcal{F})$, extends to derivations of tensor and Grassmann algebras, $p^* \in \mathrm{der}\{(\mathrm{der}\,\mathcal{F})^{*\wedge}\}$ [Cruz Guzman and Oziewicz 2003, 2006].

8.1 Definition (Electric and magnetic fields). The differential Pfaffian one-form of the electric field, $E = E(F, p)$, and differential bi-form of the magnetic field $B = B(F, p)$, relative to an observer p (measured by an observer p), are defined as follows [Fecko 1997, Kocik 1997, Cruz and Oziewicz 2003],

$$E \equiv E(F, p) \equiv i_P F, \qquad B \equiv B(F, p) \equiv i_P\{(-g^{-1}P) \wedge F\}. \tag{8.1}$$

One can check the following easy Theorem.

8.2 Theorem. *The following identity, any-observer-independence of F, is considered incorrectly in some textbooks as the definition of the electric and magnetic fields,*

$$F = B(F, p) - (g^{-1}P) \wedge E(F, p) = B' - (g^{-1}Q) \wedge E' = \dots \tag{8.2}$$

Definitions (8.1) allows to relate observations of the electric and magnetic fields made by two observers in mutual relative motion,

$$i_{\mathbf{v}} F = i_{\mathbf{v}} B + (E\mathbf{v}) g^{-1} P. \tag{8.3}$$

Most textbooks wish to avoid Grassmann algebra of multivectors and multiforms, and therefore the magnetic field is defined artificially as the pseudovector field $\mathbf{B} \in \operatorname{der}\mathcal{F}$, or as the differential pseudo-*one*-form $\mathfrak{B} \equiv g^{-1}\mathbf{B} \in (\operatorname{der}\mathcal{F})^*$, in the framework of the Gibbs vector calculus of the right-hand-rule. Josiah Willard Gibbs admits that was not able to understand Hermann Grassmann. The price is that the vector field $\mathbf{B}$ is orientation-dependent.

In three dimensions the orientation-dependent binary cross product of vectors, the vector product, $\vec{a}\,\mathbf{x}\,\vec{b}$, was invented by Clifford, popularized by Heaviside's *Electromagnetic Theory* [1893], and by Gibbs's *Vector analysis* [1901]. Eckmann in 1942 considered in arbitrary dimensions the *multi*-ary generalization. Plebański with Przanowski [1988] introduced augmented quaternion-like algebra of para-vectors and in terms of this algebra define the binary cross product of vectors in arbitrary dimensions. However the dependence of these binary cross products on orientation is lost, or it is not explicit.

Here we define the orientation-dependent binary cross product of multivectors for dimensions ≥ 3.

8.3 Definition (Gibbs's binary cross 'x' operation). A star $*$ denote the orientation-dependent Hodge star map. Let $\dim_{\mathcal{F}}(\operatorname{der}\mathcal{F}) = n \in \mathbb{N}$ be spacetime dimension, $\operatorname{grade} A = \operatorname{grade} B = k \in \mathbb{N}$, and let $\operatorname{grade} C = n - 3k$. The binary cross product, $\mathbf{x}_C$, on k-vectors is orientation-dependent and C-dependent, and is defined as follows

$$A\,\mathbf{x}_C\,B \;\equiv\; *(A \wedge B \wedge C). \tag{8.4}$$

Evidently, the same definition apply for differential multiform. This *binary* cross product does *not* exists for dimensions ≤ 2. It is well defined for one-vectors for $3 \leq \dim$, for bi-vectors for $6 \leq \dim$, etc. In particular for $\dim = 3$ the cross depends on a scalar field, $\operatorname{grade} C = 0$, and for $\dim = 4$ the cross $\mathbf{x}_C$ depends on the choice of the auxiliary vector field C, therefore in fact it is a ternary operation, and not binary. This important vector-field-dependence of the cross in *fourth* dimension is suppressed (or not realized) when writing four Maxwell differential equations and Lorentz transformations of the electric and magnetic fields.

In order to compare transformations in categorical relativity with Lorentz transformations, we need temporarily also the artificial orientation-dependent magnetic field, in spite of our deep conviction that the physical measurable magnetic field is in fact orientation-free.

8.4 Definition (Magnetic field orientation-dependent). The magnetic field as the pseudovector field, $\mathbf{B} = g\mathfrak{B} \in \operatorname{der}\mathcal{F}$, and as a pseudo-one-form $\mathfrak{B} \in (\operatorname{der}\mathcal{F})^*$, is orientation-dependent, $\mathbf{B} \equiv \mathbf{B}(F, p, *) = g\mathfrak{B} \equiv g\, i_P * F$.

8.5 Lemma. $B = i_P * \mathfrak{B}$, *and* $\mathfrak{B} = i_P * B$.

We are going to show the coordinate-free derivation of the transformations of the electric and magnetic fields in categorical relativity with binary velocity-morphism. We will show that categorical relativity gives different transformation of the electric and magnetic fields for a moving observer.

8.6 Lemma. *Let the differential one-form of the electric field,* $E \equiv E(F, p)$, *the two-form of magnetic field* B, *and a velocity vector field* $\mathbf{v} \equiv \varpi_g(P, Q)$ *of a body* q *relative to an observer* p, *are measured by an observer* p. *This imply*

$$p\,\mathbf{v} \equiv 0, \qquad i_P E \equiv 0, \qquad i_P B \equiv 0. \tag{8.5}$$

By $i_{\mathbf{v}} \in \operatorname{der}(\text{Grass})$ *we denote the graded derivation of the Grassmann algebra. Therefore* $i_{\mathbf{v}} E \equiv E\mathbf{v}$ *is a scalar field given by the evaluation of the differential one-form* E *on a vector field* $\mathbf{v}$. *Similarly* $i_{\mathbf{v}} B$ *is a Pfaffian one-form. Then*

$$\begin{aligned} q^* E &= -\gamma(i_{\mathbf{v}} E) \cdot g^{-1} Q, \qquad q^* i_{\mathbf{v}} B = 0,\\ q^* B &= +\gamma(i_{\mathbf{v}} B) \wedge g^{-1} Q. \end{aligned}$$

In [Cruz and Oziewicz 2003], we noted that for every scalar field $f \in \mathcal{F}$, there is a gauge transformation of the electric field, $E \longmapsto E + f g^{-1} P$ leaving invariant electromagnetic field $F = B - g^{-1} P \wedge E$.

The electric and magnetic fields relative to an observer q, $E' = E(F, q)$ and $B' = B(F, q)$ correspondingly, can be expressed by means of the electric and magnetic fields related to an observer p as follows.

8.7 Theorem. *Let* $\mathbf{v} \equiv \varpi(P, Q)$ *be a binary (internal) velocity of an observer* q *relative to an observer* p. *Let* E' *and* B' *be the electric and magnetic fields measured by an observer* q. *Let* E *and* B *be the electric and magnetic fields measured by an observer* p. *Then, these fields are related by means of the following expressions*

$$\begin{aligned} E' &= \gamma(E + i_{\mathbf{v}} B) \quad + \quad \gamma^2 (i_{\mathbf{v}} E) \cdot g^{-1} Q,\\ B' &= B + (g^{-1}\mathbf{v}) \wedge E + (g^{-1} Q) \wedge \{\gamma i_{\mathbf{v}} B + (\gamma - \tfrac{1}{\gamma}) E\}. \end{aligned}$$

Proof. Note that i_Q is grade -1 derivation, and q^* is grade 0 derivation. The commutator of derivations is again a derivation, and therefore we have $i_Q \circ q^* = i_Q$.

The following inference does not need neither coordinates, nor basis, nor isometric Lorentz transformation.

$$\begin{gathered} E' \equiv i_Q F = i_Q q^* F = i_Q q^*(p^* + 1 - p^*)F \\ = i_Q\{q^* p^*(p^* F) + q^*(1 - p^*)[(1 - p^*)F]\} \\ = -\gamma i_Q\{(g^{-1}Q) \wedge [i_P p^* F + i_{\mathbf{v}}(1 - p^*)F]\} \\ = -\gamma i_Q\{(g^{-1}Q) \wedge (E + i_{\mathbf{v}} B)\} \\ = \gamma(1 - q^*)(E + i_{\mathbf{v}} B) = \gamma(E + i_{\mathbf{v}} B) - \gamma q^* E. \quad \square \end{gathered}$$

Note that in adopted coordinates, $P = \partial_t$, $Q = \partial_{t'}$ and $-g^{-1}Q = dt'$.

9 Clocks, Proper time, adopted frames and metric tensor

In this note a *frame* is a synonym of a *basis* in a modul. Basis is coordinate-free concept, known as the Cartan's 'moving frame'. Physical phenomenons are frame-free and coordinate-free.

The phrase *reference frame*, is used by other authors in many many different meanings. To give an overview of different understanding of the phrase *reference frame* would need a separate paper. Few examples.

Matolcsi [1993]. The material object like room and the car, are examples of observers [1993 §3]. This is the same in the present note. A reference frame (= system) is a coordinatization of time and of space of observer. In the present note observer is coordinate-free and basis-free.

Klioner and Soffel [1998] consider *reference system* as the synonym of the coordinate chart (why another name for well understood coordinate chart?). A reference frame is a materialization relative to coordinate chart, *i.e.* a reference frame is coordinate-dependent. A coordinate frame, as used by Klioner and Soffel, is not a frame as used in the present note.

Waldyr Rodigues and Edmundo Capelas de Oliviera, in the excellent recent monograph [2005], consider frame as the synonym of the basis, however the reference frame is defined as the curve, therefore the reference frame is not a frame.

Pervushin [2005]. A reference frame is a three-dimensional coordinate basis with a watch.

Oziewicz in the present note. A frame is a basis. An observer is $(1,1)$-tensor field, an idempotent, and it is coordinate-free and bases-free. A massive body is an object of groupoid category.

9.1 Definition (Clock). Let β be differential one-form, and P be a vector field. If $\beta P \neq 0$, transversal condition, then β is said to be a P-clock. Clock is metric-independent. A Pfaff form β such that $\beta \wedge d\beta = 0$, is said to be Frobenius integrable.

P-clock is not unique. Let a differential one-form α be associated with the given vector field, *i.e.* $\alpha P = 0$. Then $(\beta + \alpha)P \neq 0$, and $\beta + \alpha$ is again another P-clock (provided that $\beta + \alpha$ is Frobenius integrable). In particular if a scalar field x is an integral for P, *i.e.* $Px = 0$, and if $\beta = dt$, then $t + x$ is again a P-clock, $P(t + x) \neq 0$, [Eckart 1940, page 920, formula (15)].

9.1 Minkowski's proper time

If c is an embedding of a one-dimensional sub-manifold, than the pull-back c^* is a morphism of algebras,

$$\begin{array}{ccc} C & \xrightarrow{c} & \text{space-time manifold } M \\ \mathcal{F}_C & \xleftarrow{c^*} & \text{algebra } \mathcal{F}_M. \end{array} \tag{9.1}$$

9.2 Definition (Integral curve). Let τ be coordinate on the one-dimensional sub-manifold C, and $\frac{d}{d\tau}$ be a vector field on C. An embedding, $c : C \hookrightarrow$ manifold, is said to be integral curve for a vector field X, if the pull-back c^* intertwine the vector fields acting (as derivations) on tensor and Grassmann algebras of differential forms. For arbitrary differential Pfaff form β we have

$$\frac{d}{d\tau} \circ c^* = c^* \circ X \quad \Longleftrightarrow \quad c^*\beta = \{(\beta X) \circ c\} d\tau. \tag{9.2}$$

In particular, $c^*(g^{-1}X) = \{X^2 \circ c\} d\tau$.

Every *not* light-like vector field X on pseudo-Riemannian manifold posses the unique metric-dependent differential Pfaff form that define the metric-dependent scalar magnitude of the integral curves of X. For time-like X this differential form is as follows,

$$\frac{-g^{-1}X}{\sqrt{-g^{-1}(X \otimes X)}}, \qquad g\text{-dependent length of } c \equiv \int_c \frac{-g^{-1}X}{\sqrt{-g^{-1}(X \otimes X)}}$$
$$= \int_C c^* \frac{-g^{-1}X}{\sqrt{-g^{-1}(X \otimes X)}} = \int_C \sqrt{-X^2 \circ c}\, d\tau = \int_C d\tau, \quad \text{if } X^2 = -1. \tag{9.3}$$

The unique differential Pfaff form of the metric-dependent Q-clock, *the Q-proper-time*, $\frac{-g^{-1}Q}{\sqrt{-g^{-1}(Q\otimes Q)}}$, Minkowski introduced implicitly in [1908 Appendix: Mechanics and relativity postulate, formula (3)] (considering this as the generalization of the Lorentz's *local time*),

$$\text{instead of:}\quad d\tau = c^*\frac{-g^{-1}Q}{\sqrt{-g^{-1}(Q\otimes Q)}}, \tag{9.4}$$

$$\text{Minkowski wrote:}\quad d\tau = \sqrt{-g^{-1}}. \tag{9.5}$$

Minkowski defined g-dependent Q-proper-time $d\tau$ in terms of the integral curve 'c_Q' of the time-like vector field Q. We identify the Minkowski Q-proper-time differential form, $-g^{-1}Q$ for $Q^2 = -1$, with the unique empirical simultaneity of the massive body $p = Q \otimes (-g^{-1}Q)$.

Minkowski in 1908 [§4: Special Lorentz transformations, before formula (20)], wrote that '(space-like) vector $\mathbf{v}/c$ uniquely define (time-like) normalized vector Q, and vice versa'. This must be understood that every *pair* of normalized time-like vectors, P and Q, determine the unique binary relative velocity-morphism, $\frac{\mathbf{v}}{c} = \frac{Q}{\gamma} - P$, such that $P\cdot\mathbf{v} = 0$, (2.9)-(4.4). However the given space-like velocity $\mathbf{v}$ can be the relative velocity among many different pairs of massive bodies. There is no one-to-one correspondence among pairs of massive bodies and relative velocities, $\frac{Q}{\gamma} - P = \frac{Q'}{\gamma} - P' = \ldots$ This non-uniqueness follows from the fact that the following system of two equations in four-dimensions for the given space-like vector $\mathbf{v}$, $P\cdot\mathbf{v} = 0$ and $P^2 = -1$, has two-parametric family of solutions.

In Minkowski's formula (20) in [1908 §4], the relative velocity $\mathbf{v}$ is clearly the *binary* velocity (4.4)-(4.8),

$$-g^{-1}Q = \gamma\left(-g^{-1}P - g^{-1}\frac{\mathbf{v}}{c}\right), \tag{9.6}$$

$$\begin{aligned} d\tau &= \{\sqrt{-g^{-1}}\}_{Minkowski} \\ &= c^*(-g^{-1}Q) = (\gamma\circ c)\left\{c^*(-g^{-1}P) - c^*\left(g^{-1}\frac{\mathbf{v}}{c}\right)\right\}. \end{aligned} \tag{9.7}$$

Here c is integral curve of the vector field Q, and not of the vector field P.

$$g^{-1} = -(dt)\otimes(dt) + \tfrac{1}{c^2}(dx)\otimes(dx) \quad \xrightarrow{c^*} \quad \{d(t\circ c)\}^2 + \tfrac{1}{c^2}\{d(x\circ c)\}^2$$

$$d(x\circ c) = w\,d(t\circ c),\quad d\,\mathrm{id} = \sqrt{1-\frac{w^2}{c^2}}d(t\circ c). \tag{9.8}$$

9.2 Clocks and proper-time as the differential forms

In the present paper, (as it is e.g. in [Llosa and Soler 2004]), for the given time-like vector field P, we consider the following differential Pfaffian forms to be conceptually *independent*:

Proper-time metric-independent. The Frobenius integrable differential form β with $\beta P \neq 0$. This is mathematical *not*-unique P-clock. No condition on $\ker\beta$. P-clock is metric-independent and simultaneity-independent.

Metric-dependent proper-time. The differential Pfaff form $-g^{-1}P$, with $(-g^{-1}P)P = 1$, is given uniquely by gravitational potential g. This differential Pfaff form of the observer, introduced by Minkowski in 1909, gives the *unique empirical simultaneity*, $\ker(-g^{-1}P)$ must be necessarily space-like. This unique metric-dependent simultaneity is called the Einstein proper time by Llosa and Soler [2004]. This proper-time-simultaneity was introduced by Minkowski in 1909. Einstein in 1905 not introduced metric explicitly.

In the rest of this paper, for simplicity, we consider 2-dimensional space-time with signature $(-,+)$ only.

9.3 Definition (Adopted co-frame). A co-frame, a basis in the module of the Pfaff forms, $\beta\wedge\alpha \neq 0$, is said to be *adopted* for a vector field P, or just P-co-frame, if $\beta P = 1$ and $\alpha P = 0$. We will denote P-co-frame using index α^p and β^p. A dual P-frame of a vector fields is $P \wedge X_p \neq 0$, where $\alpha^p X_p \equiv 1$ and $\beta^p X_p \equiv 0$. A P-frame, $P \wedge X_p \neq 0$, need not to be orthogonal.

In two-dimensions the differential form α is unique up the scalar field (up to integrating factor). For example if $\alpha = dx$, then a scalar field x is an integral for P. If $a \in \mathbb{R}$ is a non zero constant, then ax is again integral for the same vector field P, $P(ax) = 0$. However the change of coordinates, from x to ax, must not be interpreted as the *material* rod contraction or expansion.

In two-dimensions there is two-parametric family of adopted co-frames for each massive body

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset GL2, \qquad \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \longrightarrow \begin{pmatrix} \beta + f\alpha \\ g\alpha \end{pmatrix}. \tag{9.9}$$

In n-dimensions the adopted co-frame is given up to the action of the following inhomogeneous $GL(n-1)$ subgroup

$$\begin{pmatrix} 1 & * \\ 0 & GL(n-1) \end{pmatrix} \quad \subset \quad GLn. \tag{9.10}$$

An observer in the present paper is a tensor $(1,1)$-field p, and therefore is basis-free and coordinate-free. In particular time-like eigenvector P, $pP = P$, and the Minkoowski simultaneity form $-gP$, are both frame-free and coordinate-free. In P-co-frame every *non-inertial* observer p can be presented in the following way, where we must appreciate the difference among the unique Einstein-Minkowski physical simultaneity, and the many mathematical P-clocks,

$$p = P \otimes (-gP) \quad \text{with} \quad -gP = \beta + f\alpha = \beta' + f'\alpha = \dots \tag{9.11}$$

$$\ker(-gP) = \operatorname{span}\{X + (P\cdot X)P\} \quad \text{must be space-like}$$
$$\iff \quad 0 < X^2 + (P\cdot X)^2. \tag{9.12}$$

The metric tensors, g (in units s^{-2}), and g^{-1} (in s^2), need not to be diagonal in P-co-frame and dual P-frame. In particular we do not suppose that the curvature tensor field must vanish. This imply two-parametric family of metric tensors. Consider the pair of scalar fields

$$m \equiv X^2 \equiv g^{-1}(X\otimes X), \quad n \equiv g^{-1}(P\otimes X) = g^{-1}(X\otimes P). \tag{9.13}$$

The metric fields in P-co-frame are as follows,

$$g^{-1}P = -\beta + n\alpha \quad \Longrightarrow \quad g(\beta - n\alpha) = -P, \tag{9.14}$$
$$g^{-1}X = n\beta + m\alpha, \tag{9.15}$$
$$g\beta = -P + \frac{n}{m+n^2}(X + nP), \tag{9.16}$$
$$g\alpha = \frac{1}{m+n^2}(X + nP) \tag{9.17}$$

Consider the following volume form

$$\eta \equiv \beta\wedge\alpha, \qquad \eta(P\wedge X) = 1. \tag{9.18}$$

With respect to this volume form η, one can calculate the determinant of g, the scalar field, $*\eta \equiv \det_\eta g \equiv \eta(g\eta)$,

$$g\eta = g(\beta\wedge\alpha) = -\frac{1}{m+n^2}\,P\wedge X \quad \Longrightarrow \quad \det_\eta g = -\frac{1}{m+n^2}, \tag{9.19}$$
$$det_\eta g < 0 \quad \iff \quad 0 < m + n^2. \tag{9.20}$$

This can be summarized shortly as

$$\begin{aligned} P\cdot P &= -1, & \beta\cdot\beta &= m\,\det g,\\ X\cdot X &= m & \alpha\cdot\alpha &= -\det g,\\ P\cdot X &= n & \beta\cdot\alpha &= -n\,\det g. \end{aligned} \tag{9.21}$$

$$g^{-1} = -\beta \otimes \beta + m\,\alpha \otimes \alpha + n(\alpha \otimes \beta + \beta \otimes \alpha), \tag{9.22}$$

$$g = (\det_\eta g)\{m\,P \otimes P - X \otimes X - n(P \otimes X + X \otimes P)\}, \tag{9.23}$$

$$-g^{-1}P = \beta - n\alpha. \tag{9.24}$$

10 Transformations of adopted co-frames

The aim of this Section is the derivation the transformations among adopted co-frames, from $\{\beta^p, \alpha^p\}$ to $\{\beta^q, \alpha^q\}$. The co-frames are adopted for a pair of not necessarily inertial observers in mutual motion. The adopted-frame transformations are derived in terms of the binary relative velocity-morphism only (not-isometric and not-reciprocal). Therefore these transformations differs from Poincaré and Lorentz isometry. The family of transformations of the clocks of monads, include, among other, Stefan Marinov's transformation with the absolute mathematical clock [Marinov 1974, 1977].

We must distinguish among frame-free and coordinate-free concepts and concepts that depends explicitly on the choice of the co-frame and in particular a choice-dependent coordinates. All tensor fields, p and q, eigenvector fields P and Q, simultaneity differential forms, $-gP$ and $-gQ$, are frame-free. The change of co-frames can not change neither observer nor observed body, nor differential bi-form of the relativity of simultaneity $g(P \wedge Q)$. The relativity of simultaneity is the physical concept and must be independent on the mathematical choice of co-frames and coordinates.

10.1 Clarification (Which time is relative? and when?). Every observer p posses the unique empirical simultaneity differential form $-gP$, and many mathematical clocks $\{\beta^p\}$, $\ker \beta^p$ is isochronous for an observer p, it is a mathematical P-clock of an observer p. The simultaneity of an observed body q is given by a kernel of empirical differential form $\ker(-gQ)$, and not by mathematical clock β^q. We need avoid presupposing an artificial identification of the choice-free simultaneity $-gP$, with the choice-dependent mathematical clock β.

10.2 Assumptions (Two-body system). All considerations are valid for arbitrary not inertial observers, and for *variable* binary relative velocity-morphism $\mathbf{v} \equiv c\,\varpi(P, Q)$. Consider two-body system,

$$p = P \otimes (-gP) \quad \text{and} \quad q = Q \otimes (-gQ), \tag{10.1}$$

$$\frac{\mathbf{v}}{c} \equiv \varpi(P, Q) = \frac{Q}{\gamma} - P \in \ker\, gP, \quad 0 \neq \gamma \equiv -P \cdot Q, \tag{10.2}$$

$$Q \wedge P = \gamma \frac{\mathbf{v}}{c} \wedge P = \gamma\, Q \wedge \frac{\mathbf{v}^{-1}}{c}. \tag{10.3}$$

The inverse, $\mathbf{v}^{-1} \equiv c\,\varpi(Q,P) \neq -\mathbf{v}$, is observer-dependent, and it is not absolute

$$(-gP)\frac{\mathbf{v}^{-1}}{c} = -\gamma + \frac{1}{\gamma}, \quad (-gP)\mathbf{v} = 0. \tag{10.4}$$

We denote by β^q, α^q an adopted Q-co-frame, *i.e.* $\alpha^q Q \equiv 0, \beta^q Q \equiv 1$. Analogously β^p, α^p is an adopted P-co-frame. A differential form β^p is not-unique P-clock,

$$\mathbf{v} = (\beta^p\mathbf{v})P + (\alpha^p\mathbf{v})X_p = (\beta^q\mathbf{v})Q + (\alpha^q\mathbf{v})X_q, \tag{10.5}$$

$$\mathbf{v} \neq 0 \quad \Longleftrightarrow \quad \alpha^p\mathbf{v} \neq 0 \quad \& \quad \alpha^{\mathbf{v}} \neq 0. \tag{10.6}$$

The Minkowski differential form $-gP \in (\operatorname{der}\mathcal{F})^*$, is the unique empirical simultaneity differential form for an observer p. The concept of the relative velocity is frame-independent, but it is metric-dependent, needs Einstein-Minkowski simultaneity. We are going to show that the length/rod-contraction or extension, and clock-dilation and clock-retardation, are frame-dependent. Being frame-dependent are not material and not biological respectively. In contrast, the dilation of simultaneity is frame-free. Being frame-free has the objective physical meaning.

10.3 Lemma. *Let introduce the following frame-dependent scalar fields,*

$$\begin{aligned} a \equiv \alpha^q X_p, \quad h \equiv \beta^q X_p, \quad j \equiv \beta^q P = \tfrac{1}{\gamma} - \beta^q \tfrac{\mathbf{v}}{c}, \\ \tilde{a} \equiv \alpha^p X_q, \quad \tilde{h} \equiv \beta^p X_q, \quad \tilde{j} \equiv \beta^p Q = \gamma(1 + \beta^p \tfrac{\mathbf{v}}{c}). \end{aligned} \tag{10.7}$$

Then,

$$\begin{aligned} \alpha^q &= a\alpha^p - (\alpha^q \tfrac{\mathbf{v}}{c})\beta^p, \qquad & \alpha^p &= \tilde{a}\alpha^q + \gamma(\alpha^p \tfrac{\mathbf{v}}{c})\beta^q, \\ \beta^q &= j\beta^p + h\alpha^p, & \beta^p &= \tilde{j}\beta^q + \tilde{h}\alpha^q, \end{aligned} \tag{10.8}$$

$$\{aj + h\alpha^q \tfrac{\mathbf{v}}{c}\}\{\tilde{a}\tilde{j} - \gamma\tilde{h}\alpha^p \tfrac{\mathbf{v}}{c}\} = 1, \tag{10.9}$$

$$\beta^q \wedge \beta^p = h\,\alpha^p \wedge \beta^p = -\tilde{h}\,\alpha^q \wedge \beta^q, \tag{10.10}$$

$$\alpha^q \wedge \alpha^p = (\alpha^q \tfrac{\mathbf{v}}{c})\alpha^p \wedge \beta^p = \gamma(\alpha^p \tfrac{\mathbf{v}}{c})\alpha^q \wedge \beta^q, \tag{10.11}$$

$$(\alpha^q \wedge \beta^q)(\tfrac{\mathbf{v}}{c} \wedge X_p) + \tfrac{a}{\gamma} = aj + h\alpha^q \tfrac{\mathbf{v}}{c} \tag{10.12}$$

$$\begin{aligned} j + (\alpha^p \tfrac{\mathbf{v}}{c})h &= \tfrac{1}{\gamma} - (\beta^p \tfrac{\mathbf{v}}{c})(\beta^q P), \\ \tilde{j} &= \gamma\{1 + (\alpha^q \tfrac{\mathbf{v}}{c})\tilde{h} + (\beta^q \tfrac{\mathbf{v}}{c})(\beta^p Q)\}. \end{aligned} \tag{10.13}$$

Proof. From definition of the binary velocity-morphism $Q = \gamma(P + \frac{\mathbf{v}}{c})$, we have, $\alpha^q P = -\alpha^q \frac{\mathbf{v}}{c}$, $\alpha^p Q = \gamma\alpha^p \frac{\mathbf{v}}{c} = -\alpha^p \frac{\mathbf{v}^{-1}}{c}$. □

The frame-dependent scalar fields a and $\tilde{a}$ are responsible for the lenght/rod contraction/extension. This contraction is *not* the physical, not material, phenomenon, because depends on the choice of the mathematical frame. The categorical relativity, with *binary* relative velocity, does not predicts the lenght/rod *material* contraction, because of the mathematical freedom in the choice of the adopted co-frames.

For the scalar fields, $a, h, j, \tilde{a}, \tilde{h}, \tilde{j}$, subject the conditions (10.9), the co-frame $\{\beta^q, \alpha^q\}$, is Q-co-frame for a moved body Q. For example one can chose $a = 1$ and $\tilde{a} = 1$, *i.e.* no rod contraction.

This conclusion differs from Einstein's isometric more restrictive formulation in terms of the *ternary* velocities, and can be related with conclusion made by Jefimenko [1998]. Our conclusion again rise the long standing problem of the role of frames and coordinates in physics. Does Nature like the preferred frames and preferred coordinate-systems? like the Fock's harmonic coordinates?

10.4 Corollary. Consider the particular cases of the transformations of clocks (10.8), $\beta^q = j\beta^p + h\alpha^p$.

Larmor's dilation and Einstein's transformation. When $h\tilde{h} \neq 0$, this condition is frame-dependent, the clocks are relative, $\beta^q \wedge \beta^p \neq 0$ (10.10). Simultaneity must be relative, and is basis-free and coordinate-free,

$$\mathbf{v} \equiv \varpi(P, Q) \neq 0 \quad \Longleftrightarrow \quad (-gP) \wedge (-gQ) = g(P \wedge Q) \neq 0, \tag{10.14}$$

$$Q \wedge P = -\gamma \, \frac{\mathbf{v}}{c} \wedge P. \tag{10.15}$$

The Larmor's dilation of simultaneity and Einstein's transformation need empirical simultaneity, $g\beta^p = -P$ and $g\beta^q = -Q$. This imply $j = \tilde{j} = \gamma$. The simultaneity P-co-frame is, $(-gP) \wedge \alpha^p \neq 0$ and then a dual P-frame is g-orthogonal, $P \wedge X^p \neq 0$ and $P \cdot X_p = 0$. The Einstein-Minkowski simultaneity must be delated,

$$-gQ = \gamma(-gP - g\tfrac{\mathbf{v}}{c}), \quad g\mathbf{v} = (\alpha^p \mathbf{v})gX_p, \quad -gP = \gamma(-gQ - g\tfrac{\mathbf{v}^{-1}}{c}).$$

This is the same as the Lorentz transformation of simultaneity, however we are not obliged for the choice $g\alpha^p = X_p$.

Marinov's absolute clock. When $h = 0 = \tilde{h}$, clock is absolute, $\beta^q \wedge \beta^p = 0$. This imply, from (10.9), that $a\tilde{a} = 1$ and $j\tilde{j} = 1$.

Let in particular, $\beta^p \mathbf{v} = 0 = \beta^q \mathbf{v}$. Then $j = \frac{1}{\gamma}$ and $\tilde{j} = \gamma$, *i.e.* clock is retarded, $\beta^q = \frac{1}{\gamma}\beta^p$. This clock retardation is not biological, because is frame-dependent. The Marinov's transformation means that there are absolute mathematical clocks, however does *not* imply that simultaneity is absolute [Marinov 1974, 1977; Vargas and Torr 1986 p. 115].

Proper times without dilation. Can we choose the mathematical clocks, β^p and β^q, in such way that there would be neither clock-dilation and no clock-retardation? This means, can we chose $j = 1 = \tilde{j}$?

Robertson [1949], and Vargas since 1979 [Vargas 1984, ..., 2000], consider the test theory of special relativity as the family of the linear v-dependent coordinate transformations. Robertson's boost depends on three scalars a, h, j. Vargas shown, that the Ives and Stillwell optical experiment [Ives and Stillwell 1938; Vargas 1984, 1986, page 1097], agrees with the transformation of the proper times (10.8)-(10.13), if these clocks are given by Minkowski simultaneity, *i.e.* if, in an identity (10.13), $\beta^p \frac{\mathbf{v}}{c} = 0$.

Jefimenko in 1998 demonstrated that the Heaviside theory of electromagnetic retardation [Heaviside 1888], allows to derive and re-understand the most important practical electromagnetic effects without invoking the Fitzgerald length contraction [Fitzgerald 1889], and without the Larmor time dilation [Larmor 1897]. The Einstein isometric special relativity [Einstein 1905] predicts the Fitzgerald material length contraction and the Larmor biological time dilation in terms of the isometric Lorentz transformation. Jefimenko shows that the experiments interpreted as proofs of material length contraction and biological time dilation need not to be manifestations of these effects [Jefimenko 1998].

10.5 Corollary (Galileo Galilei 1632, space is relative). By definition $\ker \alpha^p$ is a set of points in a space of an observer p, $\alpha^p P = 0$. Analogously $\ker \alpha^q$ is a space of a body q, $\alpha^q Q \equiv 0$. For not vanishing relative velocity $\mathbf{v} \neq 0$, relativity of spaces is obligatory. The identity (10.11) imply

$$\alpha^q \wedge \alpha^p = 0 \quad \Longrightarrow \quad \mathbf{v} \text{ must be time-like, a contradiction,} \tag{10.16}$$

$$\mathbf{v} \neq 0 \quad \Longleftrightarrow \quad \alpha^q \wedge \alpha^p = -(\alpha^q \tfrac{\mathbf{v}}{c})\beta^p \wedge \alpha^p \neq 0. \tag{10.17}$$

11 Outline

The main conclusion: observer-independence and the Lorentz/Poincare-group-invariance are *different* concepts. Categorical relativity does not need the Lorentz group.

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