SPHERICAL STARS WITHOUT SCHWARSZCHILD SINGULARITY

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CONVENTIONS AND ABBREVIATIONS

The sign conventions for the metric and curvature tensors are (-, +, +) in the terminology of Mismer, Thorne & Wheeler [1]. That is, the metric signature is (+, -, -, -).
For this paper we use units in which c = G =1.
The following symbols and abbreviations are used throughout:

∂_μ or $_{,\mu}$	partial derivative
D_μ or $_{;\mu}$	covariant derivative
ln	natural logarithm
i, j, k,….	Latin indices equal to 1, 2 & 3
λ, μ, ν,…	Greek indices equal to 0, 1, 2, & 3
cst	constant quantity

1 – INTRODUCTION [1].

The basic hypothesis in GR is that spacetime is a four dimensional differentiable Riemannian manifold U whose the metric has the signature (+, -, -, -).

$$ds^2 = g_{\lambda\mu}\ dx_\lambda\ dx_\mu \tag{1.1}$$

There is no point in retracing the history of what led to this hypothesis; it is enough to know that essentially its source is the Mach's Principe of local equivalence which identifies the fields of gravitation and acceleration. It will later be seen that the definitions of time and space are linked to the metric ds² (but not as directly as is normally supposed).

To this framework must be added a description the material content of U; this done by bringing the stress-energy tensor $T_{\lambda\mu}$ into Einstein's equations to give the geometry of U.

$$R_{\lambda\mu} - ½\ R\ g_{\lambda\mu} = 8\pi\ T_{\lambda\mu} \tag{1.2}$$

Where $R_{\lambda\mu}$ is the Ricci tensor and $R = g^{\lambda\mu}R_{\lambda\mu}$ the Riemannian curvature.

In the external case i.e. when $T_{\lambda\mu} = 0$, the trajectories of particles are geodesics of the metric ds^2. This property has not been always verified for interior case.

2 – HOLONOMIC MEDIUMS [2].

If we assume that U contains a material distribution such that the stress-energy tensor can be written:

$$T_{\lambda\mu} = r \; u_\lambda \; u_\mu - \theta_{\lambda\mu} \tag{2.1}$$

Where:

r is a positive scalar.
u_λ is the 4- velocity of the medium.
$\theta_{\lambda\mu}$ is a symmetrical covariant tensor.

Then the distribution described by $T_{\lambda\mu}$ can be said to be a **holonomic medium** if and only if the vector K defined by:

$$r \; K_\mu = D_\lambda \; \theta^\lambda{}_\mu \tag{2.2}$$

is a gradient. Well then we take:

$$K_\lambda = \partial_\lambda \ln F \tag{2.3}$$

r being the pseudo-density and F the index of the distribution.
In that case the flow lines of the medium are geodesics of the conformal metric:

$$d\sigma^2 = F^2 \; ds^2 = \gamma_{\lambda\mu} \; dx_\lambda \; dx_\mu \tag{2.4}$$

The metric $d\sigma^2$ is thus the only one having physical reality. Consequently, the notions of time and space must be deduced from it.

We define the **vorticity tensor** of the medium by:

$$\Omega_{\lambda\mu} = \partial_\lambda (Fu_\mu) - \partial_\mu (Fu_\lambda) \tag{2.5}$$

A. Lichnerowicz says that the motion of a holonomic medium is without vortices or **irrotational** if and only if:

$$\Omega_{\lambda\mu} = 0 \tag{2.6}$$

It is well to remember that a perfect fluid of density ρ and pressure p has a stress-energy tensor:

$$T_{\lambda\mu} = (\rho + p) u_\lambda u_\mu - p g_{\lambda\mu} \quad (2.7)$$

If an equation of state **ρ = φ(p)** exits the perfect fluid is a holonomic medium with:

$$r = \rho + p \qquad F = \exp\left(\int dp/(\rho + p)\right) \quad (2.8)$$

3 – COMOVING COORDINATE SYSTEMS AND ABSOLUTE TIME [3], [4], [6], [7].

Definition. It is said that a coordinate system of U is comoving if and only if:

$$u^i = 0 \quad (3.1)$$

Hence, we have:

$$u^0 = 1/\sqrt{(g_{00})} \qquad u^\lambda = \delta^\lambda{}_0 / \sqrt{(g_{00})} \qquad u_\lambda = g_{0\lambda} / \sqrt{(g_{00})} \quad (3.2)$$

Theorem 1. Let a holonomic medium there exist a comoving coordinate system such we have:

$$d\sigma^2 = (dx^0)^2 + 2 \gamma_{0i} dx^0 dx^i + \gamma_{ij} dx^i dx^j \quad (3.3)$$

with

$$\partial_0 \gamma_{0i} = 0 \quad (3.4)$$

Proof. With the possible coordinate transformations we can choose the value of four quantities, hence it exist a comoving coordinate system such that $\gamma_{00} = 1$ i.e.

$u^1 = u^2 = u^3 = 0$ & $\gamma_{00} = 1$

We note $\Gamma^\lambda{}_{\mu\nu}$ the Christoffel symbol of $d\sigma^2$, the geodesic equation of $d\sigma^2$ is:

$$d^2x^\lambda/d\sigma^2 + \Gamma^\lambda{}_{\mu\nu} (dx^\mu/d\sigma)(dx^\nu/d\sigma) = 0 \quad (3.5)$$

The coordinates are comoving, hence the curves (x^1, x^2, x^3) =cst are geodesic i.e.

$dx^\mu/d\sigma = \delta^\mu{}_0$

(3.5) gives $\Gamma^\lambda{}_{00} = 0$ hence

$\Gamma^i{}_{00} = ½\, \gamma^{i\lambda} (\partial_0 \gamma_{0\lambda} + \partial_0 \gamma_{\lambda 0} - \partial_\lambda \gamma_{00}) = 0$

Hence

$\gamma^{ij} \partial_0 \gamma_{0j} = 0$

And

$\partial_0 \gamma_{0i} = 0$

That completes the proof.

Theorem 2. Let a holonomic medium where the motion is without vortices then:

1) It exists a comoving coordinate system such that:

$$d\sigma^2 = d\tau^2 - \eta_{ij}\, dx^i dx^j \qquad (3.6)$$

$$ds^2 = d\tau^2 / F^2 - h_{ij}\, dx^i dx^j \qquad (3.7)$$

Where h_{ij} is definite positive.

2) $r \sqrt{(h)} / F = f (x^1, x^2, x^3)$ (3.8)

Where $h = \det(h_{ij})$.

Proof.

Firstly, we apply the theorem 1 and we utilize a comoving coordinate system satisfying to (3.3) & (3.4).

$F^2 g_{00} = \gamma_{00} = 1$

$g_{00} = 1/ F^2$

We consider the vorticity tensor:

$\Omega_{\lambda\mu} = \partial_\lambda (F u_\mu) - \partial_\mu (F u_\lambda)$

$\Omega_{\lambda\mu} = \partial_\lambda (F^2 g_{0\mu}) - \partial_\mu (F^2 g_{0\lambda})$

$\Omega_{\lambda\mu} = \partial_\lambda \gamma_{0\mu} - \partial_\mu \gamma_{0\lambda}$

The movement is without vorticity hence:

$\Omega_{\lambda\mu} = 0$

Hence with (3.4)

$\partial_i \gamma_{0j} = \partial_j \gamma_{0i}$ $\qquad$ $\partial_0 \gamma_{0i} = 0$

Hence it exits a numerical function $f = f(x^1, x^2, x^3)$ such as:

$\gamma_{0i} = \partial_i f$

Let $\tau = x^0 + f$

$d\tau = dx^0 + \partial_i f\, dx^i = dx^0 + \gamma_{0i}\, dx^i$

$d\tau^2 = (dx^0)^2 + 2\gamma_{0i}\, dx^0 dx^i + \gamma_{0i}\gamma_{0j}\, dx^i dx^j$

$(dx^0)^2 + 2\gamma_{0i}\, dx^0 dx^i = d\tau^2 - \gamma_{0i}\gamma_{0j}\, dx^i dx^j$

We put in (3.3)

$d\sigma^2 = d\tau^2 + (\gamma_{ij} - \gamma_{0i}\gamma_{0j})dx^i dx^j$

Let $\eta_{ij} = \gamma_{0i}\gamma_{0j} - \gamma_{ij}$

We obtain (3.6)

$d\sigma^2 = d\tau^2 - \eta_{ij}\, dx^i dx^j$

Lastly with $h_{ij} = \eta_{ij} / F^2$ we are:

$ds^2 = d\sigma^2 / F^2 = d\tau^2 / F^2 - h_{ij}\, dx^i dx^j$

Secondly, we write the conservation identities.

$D_\lambda T^\lambda{}_\mu = 0$

$D_\lambda (r u^\lambda u_\mu) - D_\lambda \theta^\lambda{}_\mu = 0$

$D_\lambda (r u^\lambda u_\mu) – r \partial_\lambda F / F = 0$

We utilize a classical expression of the divergence of a symmetric tensor and the components of the 4-velocity.

$u^\lambda = F \delta^\lambda{}_0$ $\qquad$ & $\qquad$ $u_\lambda = \delta^0{}_\lambda / F$

$\partial_\lambda (r \delta^\lambda{}_0 \delta^0{}_\mu \sqrt{(-g)}) / \sqrt{(-g)} - ½ (\partial_\mu g_{\alpha\beta}) (r \delta^\alpha{}_0 \delta^\beta{}_0 F^2) - r \partial_\mu F / F = 0$

Where $g = \det (g_{\lambda\mu}) = h / F^2$

Therefore

$\partial_\lambda (r \delta^\lambda{}_0 \delta^0{}_\mu \sqrt{(h)} / F)F / \sqrt{(h)} - ½ (\partial_\mu g_{00}) (r F^2) - r \partial_\mu F / F = 0$

But $g_{00} = 1/ F^2$

$\partial_0 (r \delta^0{}_\mu \sqrt{(h)} / F)F / \sqrt{(h)} - ½ (-2 \partial_\mu F / F^3) (r F^2) - r \partial_\mu F / F = 0$

$\partial_0 (r \delta^0{}_\mu \sqrt{(h)} / F)F / \sqrt{(h)} = 0$

$\partial_0 (r \delta^0{}_\mu \sqrt{(h)} / F) = 0$

$\partial_0 (r \sqrt{(h)} / F) = 0$

That completes the proof.

The two theorems preceding have a important consequence.

The time τ is the same for all points of U in relative rest. Therefore this is an absolute time defined with a univocal manner.

These results are particularly applicable to a perfect fluid having an equation of state linking ρ and p, but they can be used in other cases i.e. for a scalar field interacting with dust [5].

4 – GRAVITATION FIELD WITH A SPHERICAL SYMMETRY

We consider a gravitational field having a spherical symmetry generated by an arbitrary material distribution of dust (zero pressure), having the same symmetry.

SETTING IN EQUATION

In this case we are:

$F = 1 \quad \& \quad ds^2 = d\sigma^2$

Hence there is a comoving coordinate system such as:

$$ds^2 = d\tau^2 - e^{2b} dy^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (4.1)$$

Where

$b = b(\tau, y) \quad \& \quad r = r(\tau, y)$

The Einstein field equations are:

$$r_{,\tau}{}^2 / r^2 + 2 b_{,\tau} r_{,\tau} / r - e^{-2b} (2 r_{,y\,,y} + r_{,y}{}^2 / r^2 - 2 b_{,y} r_{,y} / r) + 1 / r^2 = 8\pi\rho \qquad (4.2)$$

$$2 r_{,\tau\,,\tau} / r + r_{,\tau}{}^2 / r^2 - e^{-2b} r_{,y}{}^2 / r^2 + 1 / r^2 = 0 \qquad (4.3)$$

$$b_{,\tau,\tau} + r_{,\tau,\tau} / r + b_{,\tau} r_{,\tau} / r + b_{,\tau}^2 - e^{-2b} (r_{,y,y} / r - b_{,y} r_{,y} / r) = 0 \quad (4.4)$$

$$b_{,\tau} r_{,y} / r - r_{,\tau,y} / r = 0 \quad (4.5)$$

RESOLUTION

(4.5) gives:

$$b_{,\tau} = r_{,\tau,y} / r_{,y}$$

$$e^{2b} = r_{,y}^2 / (1 + a) \quad (4.6)$$

Where $a = a(y)$

We set in (4.3) and we obtain:

$$2 r r_{,\tau,\tau} + r_{,\tau}^2 - a = 0$$

Thus

$$r_{,\tau}^2 = a + A / r \quad (4.7)$$

Where $A = A(y)$

It is easy to integrate this differential equation and we obtain the Tolman's solutions. [8]

(4.4) don't give anything moreover, we set in (4.2) and we have:

$$A_{,y} = 8 \pi \rho r^2 r_{,y} \quad (4.8)$$

DISCUSSION

We apply the preceding result to a spherical star with a mass M and we consider the exterior case i.e. $\rho = 0$ hence (4.8) gives:

$$A_{,y} = 0$$

$$A = cst$$

(4.7) gives by derivation:

$$r_{,\tau,\tau} = - A / (2r^2) \quad (4.9)$$

If we consider a point at rest (the coordinate system is comoving), we have $y = cst$ hence:

$r_{,\tau,\tau} = d^2r / d\tau^2$

We obtain:

$d^2r / d\tau^2 = - A / (2r^2)$

It is enough to pose **A = 2M** et we obtain the Newton law.

$$d^2r / d\tau^2 = - M / r^2 \qquad (4.10)$$

A test particle of mass m is subjected to a central force being worth: $-mM / r^2$.

CONCLUSION

The metric generated by a star of mass M, with spherical symmetry, is written by using the theorems 1 & 2:

$$ds^2 = d\tau^2 - r_{,y}^2\, dy^2 / (1 + a) - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (4.10)$$

With: $r_{,y}^2 = a + 2M / r$ & $a = a(y)$

The preceding results show that the Schwarzschild solution is inadequate to describe the field creates by a spherical star. The time of Schwarzschild is not an absolute time and its use is not satisfactory. Moreover several solutions (4.10) do not have others solutions that the origin, the singularity of Schwarzschild is thus a fictitious singularity. For example if a = 0 we have the solution of Lemaître:

$$ds^2 = d\tau^2 - (2M / r)\, dy^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (4.11)$$

With $r = (9M / 2)^{1/3} (y + \tau)^{2/3}$

5 -MOTION IN A FIELD WITH SPHERICAL SYMMETRY

To determine the geodesics of the metric:

$$ds^2 = d\tau^2 - e^{2b}\, dy^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (5.1)$$

We consider the function L defined by:

$$L = d\tau^2/ds^2 - e^{2b}\, dy^2/ds^2 - r^2 (d\theta^2/ds^2 + \sin^2\theta\, d\varphi^2/ds^2) \qquad (5.2)$$

We note ′ the derivation d /ds.

$$L = \tau'^2 - e^{2b}\, y'^2 - r^2 (\theta'^2 + \sin^2\theta\, \varphi'^2) \qquad (5.3)$$

We write the Lagrange equations.

$$(\partial L/\partial q')' - \partial L/\partial q = 0 \tag{5.4}$$

With $q = \tau, y, \theta, \varphi$.

$$\tau'' - b_{,\tau}\, e^{2b}\, y'^2 - r_{,\tau}\, r\, '(\theta^2 + \sin^2\theta\, \varphi'^2) = 0 \tag{5.5}$$

$$(e^{2b}\, y')' + b_{,y}\, e^{2b}\, y'^2 + r_{,y}\, r\, '(\theta^2 + \sin^2\theta\, \varphi'^2) = 0 \tag{5.6}$$

$$(r^2\, \theta')' + r^2 \sin\theta \cos\theta\, \varphi'^2 = 0 \tag{5.7}$$

$$(r^2 \sin^2\theta\, \varphi')' = 0 \tag{5.8}$$

(5.7) admits $\theta = \pi/2$ as particular solution, what corresponds to the motions around the star in the equatorial plane.

(5.8) gives then:

$$(r^2\, \varphi')' = 0 \tag{5.9}$$

$$r^2\, \varphi' = cst \tag{5.10}$$

We obtain the generalization of **the law of areas.**

Besides by using the Schwarzschild coordinates we have:

$$ds^2 = (1 - M/r)\, dt^2 - dr^2/(1 - M/r) - r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{5.11}$$

$$ds^2 = d\tau^2 - r_{,y}^2\, dy^2 / (1 + a) - r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Therefore for $\theta = \pi/2$ the differential equation connecting r and φ is valid for the two coordinate systems.

Consequently the considerations on the advance of the perihelion of planets and on the deviation of luminous rays are valid in the coordinate system $(\tau, y, \theta, \varphi)$.

6 – CONCLUSION

In short we obtained the following results:

1) In GR, for the large class of the holonomic mediums (perfect fluids, scalar fields, etc…) we can define an absolute time in univocal way.
2) In the case of a star with spherical symmetry we find by using absolute time:
   a) The Newtonian gravitation law.
   b) The law of the areas.
   c) The advance of the perihelion of planets.
   d) The deviation of the luminous rays.

e) The disappearance of the Schwarzschild singularity. Hence the black holes are dreams which have not any physical reality.

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