Classical electrodynamics of point-like charges without divergences

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Abstract
In this paper we pay attention to the inconsistency in the derivation of the symmetric electromagnetic energy-momentum tensor for a system of charged particles from its canonical form, when the homogeneous Maxwell's equations are applied to the symmetrizing gauge transformation, while the non-homogeneous Maxwell's equations are used to obtain the motional equation. Applying the appropriate non-homogeneous Maxwell's equation to both operations, we have obtained a modified expression for the symmetric electromagnetic energy-momentum tensor. Analyzing its structure, we suggested a method of "gauge renormalization", which allows transforming the divergent terms of classical electrodynamics (infinite self-force, self-energy and self-momentum) to converging integrals. The obtained electromagnetic energy-momentum tensor has been applied to a phenomenological description of the classical electron. We found that such a description can be done in the Lorentz invariant form at the scale less than the classical radius of the electron. In particular, we explicitly determined the expressions for the mass parameters M, M_{EM} and M_P, where M, M_{EM} are the mechanical and electromagnetic masses of classical electron, correspondingly, and M_P stands for the mass parameter, associated with the "Poincaré stresses". Further, we analyzed some other principal inferences of classical electrodynamics after the "gauge renormalization". We have found that the motional equation obtained for a non-radiating charged particle does not contain its self-force. The motional equation for a radiating particle does not yield any "runaway solutions". It has been shown that the energy flux in a free electromagnetic field is guided by the Poynting vector, whereas the energy flux in a bound electromagnetic (EM) field is described by the generalized Umov's vector, defined in the paper. The problem of linear EM momentum is also examined.

1. INTRODUCTION

The problem of infinite electromagnetic mass of the electron and its infinite self-force is as old as classical electrodynamics itself [1-8]. These infinite quantities are customarily related to the incorrectness of classical approach at very small distances. As a general assertion, the latter is undoubtly true. At the same time, in our opinion the mentioned problems partially arise from inconsistent procedure applied to the derivation of the electromagnetic energy momentum (EMEM) tensor. Namely, under symmetrization of the canonical EMEM tensor for a system of charged particles the homogeneous Maxwell's equations are applied, whereas the motional equation is derived with non-homogeneous Maxwell's equations. Applying the appropriate non-homogeneous Maxwell's equations to both operations, we obtained a new form of EMEM tensor, and by means of its further "gauge renormalization" the divergent terms of classical electrodynamics were transformed into finite quantities [9]. In section 2 we derive a new EMEM tensor and carry out its gauge renormalization. Analyzing a structure of the tensor, we reveal that it describes not only the electromagnetic interaction, but also includes the energy, which holds together "the parts" of the classical electron (the "Poincaré stresses"). This finding opens a possibility for phenomenological description of the electron within classical EM theory. Following to this way, in section 3 we determine the properties of the classical electron. In section 4 we apply the new form of EMEM tensor to obtain a motional equation of charged particle and to analyze the energy-momentum conservation law. Finally, in section 5 we summarize the obtained results.

2. ELECTROMAGNETIC ENERGY-MOMENTUM TENSOR AND ITS GAUGE RENORMALIZATION

It is known that the canonical EMEM tensor has the form [5, 6]

$$T_{EM}{}^{\mu\nu} = -\frac{1}{4\pi}\partial^{\mu} A^{\gamma} F^{\nu}{}_{\gamma} + \frac{1}{16\pi} g^{\mu\nu} F_{\gamma\alpha} F^{\gamma\alpha}, \tag{1}$$

where $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ is the tensor of EM field, A^{μ} is the four-potential, $g^{\mu\nu}$ is the metric tensor, and μ, ν=0…3. In order to transform Eq. (1) into a symmetric form, the gauge transformation

$$T_{EM}{}^{\mu\nu} \to T_{EM}{}^{\mu\nu} + \partial_{\gamma}\psi^{\mu\nu\gamma} \text{ (where } \psi^{\mu\nu\gamma} = -\psi^{\mu\gamma\nu}\text{)}, \tag{2}$$

should be applied. Choosing

$$\psi^{\mu\nu}{}_{\gamma} = \frac{1}{4\pi} A^{\mu} F^{\nu}{}_{\gamma} \tag{3}$$

and writing

$$\partial^{\gamma}\psi^{\mu\nu}{}_{\gamma} = \frac{1}{4\pi}\left(\partial^{\gamma} A^{\mu}\right) F^{\nu}{}_{\gamma} + \frac{1}{4\pi} A^{\mu}\left(\partial^{\gamma} F^{\nu}{}_{\gamma}\right), \tag{4}$$

we can transform the tensor (1) to the symmetric form

$$T_{EM}{}^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\gamma} F^{\nu}{}_{\gamma} + \frac{1}{4} g^{\mu\nu} F_{\gamma\alpha} F^{\gamma\alpha}\right), \tag{5}$$

if we recognize that

$$\partial_{\gamma} F^{\nu\gamma} = 0 \tag{6}$$

(the field equation in the absence of source charges). Eq. (5) represents the conventional expression for the tensor of EM field. Hereinafter we assume an empty space-time, wherein the metric tensor is Minkowskian.

The energy-momentum tensor for matter has the form

$$T_{M}^{\mu\nu} = mc\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{d\tau}, \tag{7}$$

where m is the mass density, and τ is the proper time. Then the total energy-momentum tensor is defined as the sum of Eqs. (5) and (7):

$$T^{\mu\nu} = T_{M}{}^{\mu\nu} + T_{EM}{}^{\mu\nu} = mc\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{d\tau} + \left(-\frac{1}{4\pi} F^{\mu\gamma} F^{\nu}_{\gamma} + \frac{1}{16\pi} g^{\mu\nu} F_{\gamma\alpha} F^{\gamma\alpha}\right). \tag{8}$$

The energy-momentum conservation law requires that the four-divergence of $T^{\mu\nu}$ should vanish:

$$\partial_{\mu}\left[\left(T_{M}\right)^{\mu}{}_{\nu} + \left(T_{EM}\right)^{\mu}{}_{\nu}\right] = 0. \tag{9}$$

Using the Maxwell' equations $\partial_{\gamma} F_{\mu\nu} = -\partial_{\mu} F_{\nu\gamma} - \partial_{\nu} F_{\gamma\mu}$, and

$$\partial_{\gamma} F^{\gamma\nu} = \frac{4\pi}{c} j^{\nu} \tag{10}$$

(j^{ν} is the four-current), we find

$$\partial_{\mu}\left(T_{EM}\right)^{\mu}{}_{\nu} = -\frac{1}{c} F_{\nu\gamma} j^{\gamma}. \tag{11}$$

Inserting Eq. (11) into Eq. (9), we derive the motional equation in the form

$$mc^{2}\frac{dv_{\nu}}{dt} = F_{\nu\gamma} j^{\gamma}, \tag{12}$$

where v_{ν} is the four-velocity.

Nowadays Eqs. (1)-(12) are widely accepted. However, Eq. (12) contains, in general, self-forces of charged particle, which become infinite for the classical point-like electron. In addition, its self-energy is infinite, too.

In the paper [9] we found that the appearance of these divergent terms is related not to an intrinsic inconsistency of classical electrodynamics, but rather to erroneous derivation of the EMEM tensor: under the gauge transformation from Eq. (1) to Eq. (5) the homogeneous Maxwell equation (6) was used, while in proving the equality (11) the non-homogeneous Maxwell equation (10) was used. Correcting this error, we applied the appropriate Eq. (10) to the gauge transformation (2)-(4) and obtained the EMEM tensor for the system of *N* discrete charges in the form

$$T_{EMG}{}^{\mu\nu} = T_{(EM)\mathrm{ex}}{}^{\mu\nu} + \sum_k \underset{(k)}{T}{}_{EEM}{}^{\mu\nu} . \tag{13}$$

Here the first term in *rhs* of Eq. (13)

$$T_{(EM)\boldsymbol{ex}}{}^{\mu\nu} = \frac{1}{4\pi}\left(-\sum_{k=1}^{N} \underset{(k)}{f}{}^{\mu\gamma} \sum_{l\neq k} \underset{(l)}{f}{}^{\nu}{}_{\gamma} + \frac{1}{4} g^{\mu\nu} \sum_{k=1}^{N} \underset{(k)}{f}{}_{\gamma\alpha} \sum_{l\neq k} \underset{(l)}{f}{}^{\gamma\alpha} \right) \tag{14}$$

looks like the conventional EMEM tensor for the system of *N* charged particles, where however the terms of self-action, containing $(f_k)(f_k)$ (*k*=1…*N*), have been excluded. The second tensor in *rhs* of Eq. (13) is defined as

$$\underset{(k)}{T}{}_{EEM}{}^{\mu\nu} = \frac{1}{4\pi}\left(-\underset{(k)}{f}{}^{\mu\gamma} \underset{(k)}{f}{}^{\nu}{}_{\gamma} + \frac{1}{4} g^{\mu\nu} \underset{(k)}{f}{}_{\gamma\alpha} \underset{(k)}{f}{}^{\gamma\alpha} \right) - \frac{1}{c} \underset{(k)}{A}{}^{\mu} \underset{(k)}{j}{}^{\nu} , \tag{15}$$

which describes only the properties of particle *k*, but not its interaction with other particles. That is why we named it as the Eigen ElectroMagnetic (EEM) energy-momentum tensor of charged particle. A reader interesting in details of derivation of Eqs. (13)-(15) is addressed to the original paper [9].

Using the tensor (13) and taking into account the matter tensor (7), we write the total energy-momentum tensor as

$$T^{\mu\nu} = \left(\sum_{k=1}^{N} \underset{(k)}{m} c \frac{d \underset{(k)}{x}{}^{\mu}}{dt} \frac{d \underset{(k)}{x}{}^{\nu}}{d\tau} + \underset{(k)}{T}{}_{EEM}{}^{\mu\nu} \right) + T_{(EM)\mathrm{ex}}{}^{\mu\nu} . \tag{16}$$

In this equation the EEM tensor (15) represents the difference of two divergent terms and, in fact, is uncertain. In these conditions we can carry out its suitable gauge modification, in order to deal with only finite quantities. For this purpose we have introduced in [9] the tensor

$$T_{mass}{}^{\mu\nu} = m_{ad} c \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{d\tau} , \tag{17}$$

named as the tensor of electromagnetic mass, and have shown that the EEM tensor $\underset{(EEM)}{T}{}^{\mu\nu}$ and $T_{mass}{}^{\mu\nu}$ are related by the gauge transformation (2). Hence we can replace $\underset{(EEM)}{T}{}^{\mu\nu}$ by $T_{mass}{}^{\mu\nu}$ in Eq. (13), and obtain the expression for EMEM tensor as follows:

$$T_{EMG}{}^{\mu\nu} = T_{(EM)\boldsymbol{ex}}{}^{\mu\nu} + \sum_k \underset{(k)}{T}{}_{mass}{}^{\mu\nu} . \tag{18}$$

Then the total energy-momentum tensor acquires the form:

$$T^{\mu\nu} = \left(\sum_{k=1}^{N} c \left[\underset{(k)}{m} + \underset{(k)}{m}{}_{ad} \right] \frac{d \underset{(k)}{x}{}^{\mu}}{dt} \frac{d \underset{(k)}{x}{}^{\nu}}{d\tau} \right) + T_{(EM)\boldsymbol{ex}}{}^{\mu\nu} . \tag{19}$$

In ref. [9] we conditionally named the parameter m_{ad} as the electromagnetic mass density, although we will see below that its physical meaning is more complicated. That is why now we use the subscript "ad", renaming this mass parameter as the "additional mass density". In order to find an explicit expression for m_{ad} of an isolated charged particle, we take into account that the gauge transformations do not influence the total energy and momentum [11, 12], and

$$\int_V T_{mass}{}^{00} dV = \int_V T_{EEM}{}^{00} dV \; ; \; \int_V T_{mass}{}^{0i} dV = \int_V T_{EEM}{}^{0i} dV \qquad (20), (21)$$

(the integration is carried out over the whole 3-space *V*). Then the straightforward calculations give

$$\int_V m_{ad} dV = M_{ad} = \frac{1}{c^2}\int_V \frac{E_s{}^2}{8\pi} dV - \frac{1}{c^2}\int_V \rho_s \varphi_s dV \,, \qquad (22)$$

(in a rest frame of the electron)

$$\gamma\vec{v}\int_V m_{ad} dV = \gamma M_{ad}\vec{v} = \int_V \frac{1}{4\pi c}\left(\vec{E}_s \times \vec{B}_s\right) dV - \vec{v}\int_V \frac{\rho_s \varphi_s}{c^2} dV \qquad (23)$$

(for a moving electron).

The subscript "s" denoted the self-fields of the electron, ρ_s is its charge density, and φ_s is the scalar potential. These equations state that the difference of two divergent integrals in their *rhs* must be finite and equal to the additional mass of particle (Eq. (22)) and additional momentum of particle (Eq. (23)). In the next section we apply Eqs. (22) and (23) for phenomenological description of the classical electron.

3. CLASSICAL ELECTRON AND "POINCARÉ STRESSES"

Looking closer at Eqs. (22), (23), we emphasize the presence of a negative terms in their *rhs*, which indicates that the energy of forces of attraction between the "parts" of classical electron has already been included into the additional mass M_{ad}. These forces are non-electromagnetic in their nature and hence they can be associated with the "Poincaré stresses". Thus, we arrive at the important conclusion: the EMEM tensor (18) describes not only electromagnetic interaction of charged particles, but it includes the negative energy, which holds together the "parts" of classical point-like electron. This interpretation of Eqs. (22), (23) allows presenting the additional mass as the sum

$$M_{ad} = M_{EM} + M_P \,, \qquad (24)$$

where

$$M_{EM} = \frac{1}{c^2}\int_V \frac{E_s{}^2}{8\pi} dV \qquad (25)$$

is the electromagnetic mass of the classical electron, and

$$M_P = -\frac{1}{c^2}\int_V \rho_s \varphi_s dV \qquad (26)$$

is the negative mass parameter, associated with the energy of "Poincaré stresses". The obtained Eqs. (22)-(26) give a tool for phenomenological description of the classical electron.

First consider Eq. (22), which is valid in the rest frame of the electron. In this frame the static electric field can be written as

$$\vec{E}_s = -\nabla \varphi_s \,. \qquad (27)$$

Then

$$\int_V \frac{E_s{}^2}{8\pi} dV = -\frac{1}{8\pi}\int_V \vec{E}_s \cdot \nabla \varphi_s dV = -\frac{1}{8\pi}\int_V \nabla \cdot \left(\vec{E}_s \varphi_s\right) dV + \frac{1}{8\pi}\int_V \varphi_s \left(\nabla \cdot \vec{E}_s\right) dV \,. \qquad (28)$$

The first integral in *rhs* of (28) is transformed into the area integral due to the Gauss theorem. Hence it is equal to zero, as far as the fields are vanished at infinity. In the second integral we use the Maxwell equation $\nabla \cdot \vec{E}_s = 4\pi\rho_s$, which yields:

$$\int_V \frac{E_s^{\ 2}}{8\pi} dV = \frac{1}{2} \int_V \rho_s \varphi_s dV . \qquad (29)$$

Inserting this equality into Eq. (22), we obtain

$$M_{ad} = -\frac{1}{2c^2} \int_V \rho_s \varphi_s dV , \qquad (30)$$

which is an infinite value. A single way to avoid such a divergent solution is to assume that the charge density of the electron ρ_s represents a discontinuous function. In other words, we assume that the classical electron has a finite radius r_0, and inside the electron ($r \le r_0$), the charge density represents some non-vanished function of r[1]: $\rho_s = \rho_s(r)$, while outside the electron ($r > r_0$), $\rho_s(r) = 0$. Then for such discontinuous function the identities of vector analysis are not, in general, applicable, and we have to compute separately the integrals of Eq. (22) inside and outside the electron. Hereinafter we assume that the Gauss theorem is applicable to any continuous functions at any distances. Then outside the electron ($r > r_0$), where its change density is equal to zero, the scalar potential and electric field are determined by the conventional equations

$$\varphi = q/r , \ E = q/r^2 . \qquad (31), (32)$$

Hence in the non-relativistic limit ($\gamma \approx 1$) Eqs. (22), (23) acquire the form

$$M_{ad} = \frac{1}{c^2} \int_{r>r_0} \frac{E_s^{\ 2}}{8\pi} dV + \frac{1}{c^2} \int_{r\le r_0} \frac{E_s^{\ 2}}{8\pi} dV - \frac{1}{c^2} \int_{r\le r_0} \rho_s \varphi_s dV = \frac{q^2}{2r_0} + \frac{1}{c^2} \int_{r\le r_0} \frac{E_s^{\ 2}}{8\pi} dV - \frac{1}{c^2} \int_{r\le r_0} \rho_s \varphi_s dV ,$$

(33)

(q being the charge of the electron),

$$\left(M_{ad}\right)_v \vec{v} = \int_{r>r_0} \frac{1}{4\pi c} \left(\vec{E}_s \times \vec{B}_s\right) dV + \int_{r\le r_0} \frac{1}{4\pi c} \left(\vec{E}_s \times \vec{B}_s\right) dV - \vec{v} \int_{r\le r_0} \frac{\rho_s \varphi_s}{c^2} dV =$$

$$= \frac{2q^2 \vec{v}}{3r_b c^2} + \frac{2\vec{v}}{3} \int_{r\le r_0} E^2 r^2 dr - \vec{v} \int_{r\le r_0} \frac{\rho_s \varphi_s}{c^2} dV. \qquad (34)$$

Manipulating with the latter equations, we have used the known results [13]

$$\int_{r_b}^{\infty} \frac{E^2}{8\pi} dV = \frac{q^2}{2r_b} , \ \int_{r_b}^{\infty} \frac{1}{4\pi} \left(\vec{E} \times \vec{B}\right) dV = \frac{2\vec{v}}{3c^2} \int_{r_b}^{\infty} E^2 r^2 dr = \frac{2q^2 \vec{v}}{3r_b c^2}$$

(hereinafter we drop the subscript "s"). In Eqs. (33) and (34) we distinguish the masses M_{ad} and $\left(M_{ad}\right)_v$ determined through the energy and momentum conservation laws, correspondingly. In the conventional classical electrodynamics their ratio is equal to 4/3, which represents the old puzzling of this theory. We see that within our approach the ratio of the electron masses determining through its energy (M_{ad}) and linear momentum ($\left(M_{ad}\right)_v$) depends on the contribution of other terms of Eqs. (33) and (34). In order to evaluate these terms, we apply the Maxwell's equation and the Gauss theorem to the space region inside the classical electron, $r \le r_0$. Then we obtain:

$$E(r) = \frac{4\pi}{r^2} \int_0^r \rho(r) r^2 dr , \qquad (35)$$

and $\varphi(r) = 4\pi \int_0^r \frac{1}{r^2} \left[\int_0^r \rho(r) r^2 dr \right] dr .$ (36)

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[1] Due to the isotropy of space, the charge density cannot depend on the vector $\vec{r}$.

Combining Eqs. (33)-(36), we get

$$M_{ad}c^2=\frac{q^2}{2r_0}+\int_{r<r_0}\frac{(4\pi)^2}{r^4}\left[\int_0^r\rho(r)r^2dr\right]^2dV-\int_{r<r_0}\rho(r)\int_0^r\frac{4\pi}{r^2}\left[\int_0^r\rho(r)r^2dr\right]drdV\,, \tag{37}$$

$$(M_{ad})_v c^2=\frac{2q^2}{3r_0}+\frac{2}{3}\int_{r<r_0}\frac{(4\pi)^2}{r^2}\left(\int_0^r\rho(r)r^2dr\right)^2dr-\int_{r<r_b}\rho(r)\int_0^r\frac{4\pi}{r^2}\left[\int_0^r\rho(r)r^2dr\right]drdV\,. \tag{38}$$

Thus, the ratio of *rhs* of Eqs. (37), (38) depends on the shape of the function $\rho=\rho(r)$. At first glance, we are free to construct various classical models of the electron, choosing different functions of $\rho(r)$. However, it is not the case. Namely, one can establish an essential restriction to the choice of functions $\rho=\rho(r)$, if we involve a requirement of stability of the classical electron. This requirement yields the inequality $|M_{EM}|\le|M_P|$. Further we assume that Nature acts rationally, and the energy of the "Poincaré stresses" is just sufficient to hold together the charged "parts" of classical electron. This condition signifies an exact equality $M_{EM}=-M_P$. Hence $M_{ad}=0$ (see, Eq. (24)). Then Eq. (37) is transformed to the integral equation

$$\frac{q^2}{2r_0}+\int_{r<r_0}\frac{2\pi}{r^4}\left[\int_0^r\rho(r)r^2dr\right]^2dV-\int_{r<r_0}\rho(r)\int_0^r\frac{4\pi}{r^2}\left[\int_0^r\rho(r)r^2dr\right]drdV=0 \tag{39}$$

with respect to the function $\rho(r)$. We seek a solution of this equation in the form $\rho=const\cdot r^n$ (*n* being some number), with the condition of normalization

$$\int_{V(r<r_0)}\rho dV=q\,. \tag{40}$$

The latter condition is implemented for *n*>-2, and yields

$$\rho=\frac{q(n+3)}{4\pi r_0^{\,n+3}r}\,. \tag{41}$$

Then straightforward calculations lead to the following equation with respect to *n*:

$$\frac{q^2}{2r_b}-\frac{q^2}{2r_b}\frac{(n+4)}{(n+2)(2n+5)}=0\,.$$

This square equation has two roots: *n*=-1, and *n*=-3. The latter solution does not satisfy the condition of normalization (40) and should be rejected. Thus, we get a single solution *n*=-1, which signifies that the electron's charge density falls off as 1/*r* with increase of *r* up to the boundary value $r=r_0$:

$$\rho=\frac{q}{2\pi r_0^{\,2}r}\,. \tag{42}$$

Such is the first general property of the classical electron. An insertion of Eq. (42) into Eq. (35) allows finding the electric field inside the electron. This field is constant for $r\le r_0$ and equal to

$$E=\frac{q}{r_0^{\,2}}\,.$$

Notice that it is the same, like in the Born and Infeld model of classical electron [4].

The electromagnetic mass of the classical electron is equal to

$$M_{EM}=\frac{q^2}{2r_0c^2}+\int_{r<r_0}\frac{2\pi}{c^2r^4}\left[\int_0^r\rho(r)r^2dr\right]^2dV\,. \tag{43}$$

Combining Eqs. (42) and (43), one gets:

$$M_{EM} = \frac{2q^2}{3r_0 c^2}. \tag{44}$$

A fraction of electromagnetic mass convicted inside the electron is

$$\left(M_{EM}\right)_{in} = \int\limits_{r<r_0} \frac{2\pi}{c^2 r^4} \left[\int\limits_0^r \rho(r) r^2 dr \right]^2 dV = \frac{q^2}{6r_0 c^2}, \tag{45}$$

which is equal to a quarter of the full electromagnetic mass (compare Eqs. (44) and (45)). We add that due to the equality $M_{EM} = -M_P$, the mass associated with the “Poincaré stresses” is

$$M_P = -\frac{2q^2}{3r_0 c^2}. \tag{46}$$

Further on, substituting Eq. (42) into (38), we obtain after simple calculations

$$\left(M_{ad}\right)_v = \frac{2q^2}{9r_0 c^2}. \tag{47}$$

Thus, like in conventional electrodynamics, we get a paradoxical result: on the one hand, we demanded the equality $M_{EM} = -M_P$, which yields $M_{ad} = 0$; on the other hand, Eq. (47) shows that $\left(M_{ad}\right)_v$ determined through the linear momentum of classical electron, has a finite value. This paradox represents a variety of the “4/3 puzzling”, and its resolution is similar. Namely, following to Poincaré, we have to recognize that the tensor (19) does not represent the total energy-momentum tensor of an isolated system “classical electron plus its electromagnetic field”; the internal electron’s energy of the “Poincaré stresses” must be included. Developing this idea, we replace the matter tensor for an isolated electron as an unstructured particle

$$T_M{}^{\mu\nu} = c\left[m + m_{ad}\right] \frac{dx^\mu}{dt} \frac{dx^\nu}{d\tau} \tag{48}$$

(see, Eq. (19)) by a matter tensor of fluid medium. In the rest frame of the electron this tensor has the diagonal form [6]:

$$T_M{}^{\mu\nu} = \begin{bmatrix} \left(m + m_{ad}\right)c^2 & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ 0 & 0 & p_{22} & 0 \\ 0 & 0 & 0 & p_{33} \end{bmatrix} = \begin{bmatrix} \left(m + m_{ad}\right)c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{49}$$

where the parameter p describes the “Poincaré stresses”. We have taken into account that the spatial diagonal elements are equal to each other, $p_{11} = p_{22} = p_{33} \equiv p$, due to the Pascal law. Hereinafter we adopt that the scalar pressure p is a constant value inside the electron.

For any symmetric tensor $T^{\mu\nu}$ its time-like components are transformed as (see, the problem 1 after paragraph 6 of [6])

$$T^{00} = \gamma^2 \left(T'^{00} + 2\frac{v}{c} T'^{01} + \frac{v^2}{c^2} T'^{11} \right), \tag{50a}$$

$$T^{01} = \gamma^2 \left(T'^{01} \left(1 + v^2/c^2\right) + \frac{v}{c} T'^{00} + \frac{v}{c} T'^{11} \right), \tag{50b}$$

$$T^{02} = \gamma \left(T'^{02} + \frac{v}{c} T'^{12} \right), \tag{50c}$$

$$T^{03} = \gamma \left(T'^{03} + \frac{v}{c} T'^{13} \right) \tag{50d}$$

under relative motion of two inertial reference frames K and K' at the constant velocity v along the axis x. Hence for moving electron, when the axis x coincides with the direction of $\vec{v}$, we obtain from Eqs. (49), (50) the time-like elements of the matter tensor as follows:

$$T^{00} = \gamma^2 \left((m + m_{ad})c^2 + \frac{v^2}{c^2} p \right) \approx (m + m_{ad})c^2 , \quad (51)$$

$$T^{01} = \gamma^2 \left((m + m_{ad})cv + \frac{v}{c} p \right) \approx (m + m_{ad})cv + \frac{v}{c} p , \; T^{02} = T^{03} = 0 \quad (52)$$

in the non-relativistic limit. Therefore, the mechanical momentum of moving electron is

$$P_M = \frac{1}{c} \int_V \left((m + m_{ad})cv + \frac{v}{c} p \right) dV = \int_V mv dV + \int_{V_e} \frac{v}{c^2} p dV = Mv + \frac{v}{c^2} p V_e , \quad (53)$$

V_e being the volume of classical electron. Here we take into account that p=0 outside the classical electron, and $\int_V m_{ad} dV = M_{ad} = 0$. Due to the latter condition, Eq. (53) defines the total momentum of moving electron. On the other hand, this total momentum represents the sum of Mv and the *rhs* of Eq. (47), multiplied by v. From there we find the parameter p:

$$\frac{v}{c^2} pV = \frac{2q^2 v}{9 r_0 c^2} , \; p = \frac{1}{V_e} \frac{2q^2}{9 r_0} . \quad (54)$$

The parameter r_0 in Eq. (54) has the order of magnitude of classical radius of the electron ($\approx 10^{-13}$ cm). Then the velocities of "parts" of the classical electron are ultra-relativistic due to the uncertainty relation. In this limit [6]

$$(m + m_{ad})c^2 = 3p = \frac{1}{V_e} \frac{2q^2}{3 r_0} . \quad (55)$$

Integrating the latter equation over the volume of the classical electron and taking into account the equalities $\int_{V_e} m dV = M$, $\int_{V_e} m_{ad} dV = (M_{EM})_{in} + M_P$, we further obtain:

$$M + (M_{EM})_{in} + M_P = \frac{2q^2}{3 r_0 c^2} . \quad (56)$$

Substituting the expressions for $(M_{EM})_{in}$ (Eq. (45)) and M_P (Eq. (46)) into Eq. (56), we derive an expression for the mechanical mass of classical electron:

$$M = \frac{7q^2}{6 r_0 c^2} . \quad (57)$$

As a result, we not only resolve the paradox with the masses of classical electron, obtained via its energy and linear momentum, but also get the values of the mass parameters M, M_{EM} and M_P. We point out that due to the adopted equality M_{EM}=-M_P, an observed electron's mass coincides with its mechanical mass (57).

Thus, using the gauge renormalized EMEM tensor, we have advanced the idea by Poincaré at a quantitative level and got a full phenomenological description of the classical electron.

4. CLASSICAL ELECTRODYNAMICS AFTER "GAUGE RENORMALIZATION": BASIC POINTS

In this section we will consider a motional equation derived from the equality $\partial_\mu T^{\mu\nu} = 0$, as well as the energy balance equation $\partial_\mu T^{\mu 0} = 0$ and momentum of EM field $T^{\mu 0}$, when the gauge normalized total energy-momentum tensors (19) is applied.

4.1. Motional equation for a charged particle

The motional equation is derived from the equality

$$\partial_\mu T^{\mu\nu} = 0, \tag{58}$$

If a particle does not radiate, we insert the tensor (19) into the conservation law (58). Then we obtain

$$\partial_\mu T^{\mu\nu} = \sum_k \left(c\partial_\mu \left[\left(\underset{(k)}{m} + \underset{(k)ad}{m} \right) \frac{d \underset{(k)}{x}^\mu}{dt} \right] v_l^{\ \nu} + c\left(\underset{(k)}{m} + \underset{(k)ad}{m} \right) \frac{dv_{k\nu}}{dt} + \left(\partial_\mu T_{EM(\boldsymbol{ex})}{}^{\mu\nu} \right) \right) = 0. \tag{59}$$

The latter equation is implemented, if and only if

$$\left(\underset{(k)}{m} + \underset{(k)ad}{m} \right) \frac{dv_{l\nu}}{dt} = \frac{1}{c^2} \left(F_{ex} \right)_{\nu\gamma} j_k^{\ \gamma}, \text{ and } \partial_\mu \left(\left(\underset{(k)}{m} + \underset{(k)ad}{m} \right) \frac{dx_k^{\ \mu}}{dt} \right) = 0. \tag{60), (61}$$

Now consider the motion of a single non-radiating charged particle q with the mechanical rest mass M in an external EM field. Proceeding from continuous to discrete distributions of masses and charges, and taking into account that M_{ad} =0, we obtain from Eq. (60)

$$M \frac{dv_\nu}{dt} = \frac{q}{c^2} \left(F_{\nu\gamma} \right)_{\boldsymbol{ex}} v^\gamma, \tag{62}$$

Eq. (62) has an essential difference from the conventional motional equation (12): it shows that a particle experiences the forces only due to the external EM fields, and a self-action is impossible. This result reflects our original exclusion of self-action from the electromagnetic energy-momentum tensor under the “gauge renormalization”.

When a particle radiates, we divide the tensor of electromagnetic field $F^{\mu\gamma}$ into non-radiative (bound) and radiative (free) parts: $F^{\mu\gamma} = \left(F^{\mu\gamma} \right)_{bound} + \left(F^{\mu\gamma} \right)_{free}$. Then the straightforward calculations performed in [9] give the following expression for the force of radiation reaction [9]:

$$\vec{F}_{\mathrm{r}} = -q\nabla\left(\varphi_{\mathrm{r}} - \left(\vec{v} \cdot \vec{A}_{\mathrm{r}} \right)/c \right) = -q\left(\nabla \varphi'_{\mathrm{r}} \right)/\gamma, \tag{63}$$

where φ'_{r} is the scalar potential of EM radiation in the rest frame of particle. Note that $\nabla\varphi'_{\mathrm{r}}$ and φ'_{r} have the same sign, because the electric field of EM radiation falls as $1/r$. Hence no “runaway solutions”, like a self-acceleration of radiating particle, is appeared.

4.2. Energy flux in free and bound electromagnetic fields

First consider a free EM field in the absence of charged particles. Then the electromagnetic energy-momentum tensor (19) takes its usual form (5), and the equality $\partial_\mu T^{\mu 0}$ =0 yields $\frac{\partial u}{\partial t} + \nabla\vec{S} = 0$, where the Poynting vector $\vec{S} = \frac{1}{4\pi c}\left(\vec{E} \times \vec{B} \right)$. If the EM radiation falls on a system of charged particles, then the latter equation transforms to the known equation

$$\frac{\partial u}{\partial t} + \nabla\vec{S} + \vec{j} \cdot \vec{E} = 0.$$

Hence the energy flux density remains unmodified in the classical electrodynamics after the gauge renormalization.

Now let us determine the energy balance equation for a bound EM field with the total energy-momentum tensor (19). The equality $\partial_\mu T^{\mu 0} = 0$ yields:

$$\left(\vec{j}\vec{E} \right)_{\mathrm{ex}} + \frac{\partial u_{\mathrm{ex}}}{c\partial t} + \nabla \cdot S_{\mathrm{ex}} = 0, \tag{64}$$

where $\left(\vec{j}\vec{E}\right)_{\text{ex}} = \left(F_{0\gamma}\right)_{\text{ex}} j^{\gamma}$ is the time rate of work done (without the self-forces), $u_{ex} = \left(-F^{0\gamma}F_{\gamma}^{0}/4\pi + F_{\gamma\alpha}F^{\gamma\alpha}/4\right)_{ex}$ is the part of energy density of EM field, where the "self-action" components $\vec{E}_l\vec{E}_l$ and $\vec{B}_l\vec{B}_l$ are excluded, and $\vec{S}_{\text{ex}}$ is the portion of Poynting vector, where the "self-action" components $\vec{E}_l \times \vec{B}_l$ are also excluded. It is given by the equation $S^{i}{}_{\text{ex}} = c\left(-F^{i\gamma}F_{\gamma}^{0}\right)_{\text{ex}}/4\pi$.

Eq. (64) does not yet determine the total flow of energy in a bound EM field, because the flow of EM masses should be added. Due to the fixed ratio of mechanical to EM mass, the continuity equation (61) is separately valid for the density of EM mass u_s/c^2:

$$\partial_{\mu}\left(u_s \frac{dx_s{}^{\mu}}{dt}\right) = 0, \tag{65}$$

and the total flow of EM energy is determined by summing up of Eqs. (64) and (65). Then simple, but extensive calculations performed in [9], give for the system of N non-radiating charged particles:

$$\frac{\partial u_{\Sigma}}{\partial t} + \nabla_{-\Sigma} \cdot \vec{U}_G = 0, \tag{66}$$

where we have introduced the vector

$$\vec{U}_G = \sum_{k=1}^{N} \vec{v}_k \left(\vec{E}_{\Sigma} \cdot \vec{E}_k + \vec{B}_{\Sigma} \cdot \vec{B}_k\right), \tag{67}$$

named as the generalized Umov's vector. Here $\vec{E}_{\Sigma} = \sum_k \vec{E}_k$ and $\vec{B}_{\Sigma} = \sum_k \vec{B}_k$ are the resultant electric and magnetic fields. The operator $\nabla_{-\Sigma}$ acts only on $\vec{E}_k$, $\vec{B}_k$, but not on the resultant fields.

Thus, we have got the energy balance equation (66), which determines the energy flux in a bound EM field. We see that it does not contain the term of dissipation of EM energy $\vec{j} \cdot \vec{E}$. In this connection we mention that the term $\vec{j} \cdot \vec{E}$ describes a time derivative of the kinetic energy of particles, which is equal with the opposite sign to the time rate of change of potential energy of particles in the bound EM field. In turn, the change of potential energy is already included in the partial time derivative $\partial u/\partial t$. Hence, in comparison with the energy balance equation for free EM field, the term $\vec{j} \cdot \vec{E}$ does not appear for the bound fields. This reflects a known fact that the non-radiative EM field is not absorbed by charged particles. Inasmuch as Eq. (66) represents the sum of Eqs. (64), (65), it incorporates two different effects: the flow of EM masses of all individual particles, as well as the flow of superposed bound EM fields of the particles. We notice that in the particular case, where the instantaneous velocities of all particles are equal to each other ($\vec{v}_k = \vec{v}$ for any k), Eq. (66) acquires the form

$$\frac{\partial u_{\Sigma}}{\partial t} + \nabla \cdot \left(\vec{v} u_{\Sigma}\right) = 0. \tag{68}$$

This equation shows that the resultant EM field rigidly moves together with the source particles. It is interesting that each individual particle carries its EM mass independently of other particles, but the superposition of bound EM fields from all particles transforms the sum of these individual motions into a common motion of the resultant bound EM field at the same velocity $\vec{v}$.

The results obtained in this sub-section indicate that free and bound EM fields have substantially different physical properties. This result is not surprising, because a free EM field represents an isolated system and obeys the homogeneous Maxwell's equations. In contrast, the bound EM field represents an open system [5] and is subjected to the non-homogeneous Maxwell's equations.

4.3. The momentum of free and bound EM fields

The momentum density of the EM field is the component ${T_{EM}}^{i0}/c$ (i=1…3) in the EMEM tensor. For electromagnetic radiation it is written in the known form $\vec{p} = \vec{E} \times \vec{B}/4\pi c$. For a bound EM field we determine the EMEM tensor as

$$ {T_{EM}}^{\mu\nu} = \sum_k \underset{(k)ad}{m} \frac{d\underset{(k)}{x}^{\mu}}{dt} \frac{d\underset{(k)}{x}^{\nu}}{d\tau} + \left(-F^{\mu\gamma} F_{\gamma}^{\nu} /4\pi + \frac{1}{4} g^{\mu\nu} F_{\gamma\alpha} F^{\gamma\alpha} /4 \right)_{\boldsymbol{ex}} . $$

which is derived from the tensor (19) by the exclusion of its mechanical part. Then the momentum density as a function of velocities of particles and their EM fields is

$$ \vec{p} = \sum_k \left[\vec{v}_k \gamma_k \left({E_k}^2 + {B_k}^2 \right) \right] / c^2 + \sum_{k \neq k'} \vec{E}_k \times \vec{B}_{k'} /4\pi . \tag{69} $$

The total momentum of a bound EM field is computed by integration of Eq. (69) over the 3-space:

$$ \vec{P}_{EM} = \int_V \sum_k \left[\vec{v}_k \gamma_k \left({E_k}^2 + {B_k}^2 \right) \right] / c^2 dV + \int_V \sum_{k \neq k'} \left(\vec{E}_k \times \vec{B}_{k'} /4\pi \right) dV . \tag{70} $$

It consists of two parts: the momentum density, associated with the EM mass of charged particles, and the momentum density, resulting from the superposition of EM fields of different particles. We emphasize that the first term in *rhs* of Eq. (69) represents the sum of contributions of EM momenta of the particles, associated with their EM mass, to the total momentum of that particles. Therefore, the time rate of the first term in *rhs* of Eq. (70) is rather the consequence than the cause of the force experienced by the particles. Hence the external forces, acting on charged particles, are determined by the time rate of the second term in *rhs* of Eq. (70).

Let us consider an isolated system, consisting of two non-radiating charged particles q_1 and q_2, and determine a total force exerted on this system. In general, it does not vanish, owing to violation of Newton's third law in EM interactions. Adding the mechanical momenta of both particles to Eq. (70), we obtain

$$ \vec{P}_{EM} = \gamma_1 \left(M_1 + M_{EM1} \right) \vec{v}_1 + \gamma_2 \left(M_2 + M_{EM2} \right) \vec{v}_2 + \int_V \left(\vec{E}_1 \times \vec{B}_2 + \vec{E}_2 \times \vec{B}_1 \right) dV . $$

The resulting force, acting on the particles, is

$$ \vec{F} = \frac{d}{dt} \left[\gamma_1 \left(M_1 + M_{EM1} \right) \vec{v}_1 + \gamma_2 \left(M_2 + M_{EM2} \right) \vec{v}_2 \right] = -\frac{d}{dt} \int_V \left(\vec{E}_1 \times \vec{B}_2 + \vec{E}_2 \times \vec{B}_1 \right) dV . \tag{71} $$

If the non-radiating particles are non-relativistic, then [14]

$$ \int_V \left(\vec{E}_1 \times \vec{B}_2 \right) dV = q_1 \vec{A}_{21} / c , \quad \int_V \left(\vec{E}_2 \times \vec{B}_1 \right) dV = q_2 \vec{A}_{12} / c , $$

where $\vec{A}_{21}$ is the vector potential produced by the particle 2 at the location of particle 1, and $\vec{A}_{12}$ is the vector potential of particle 1 at the location of particle 2. Hence

$$ \vec{F} = \frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = -\frac{q_1}{c} \frac{d\vec{A}_{21}}{dt} - \frac{q_2}{c} \frac{d\vec{A}_{12}}{dt} . \tag{72} $$

This equation reflects the law of conservation of the canonical momentum

$$ \vec{P}_C = \left(\vec{P}_1 + q_1 \vec{A}_{21} / c \right) + \left(\vec{P}_2 + q_2 \vec{A}_{12} / c \right) = \text{const} $$

for the considered non-radiating non-relativistic system. Eq. (72) has also been derived in [15] within the Lagrangian formalism.

Without the "gauge renormalization", the conventional Poynting vector would determine the resultant force:

$$ \vec{F} = \frac{d}{dt} \left(\gamma_1 M_1 \vec{v}_1 + \gamma_2 M_2 \vec{v}_2 \right) = -\frac{d}{dt} \int_V \left(\vec{E}_1 \times \vec{B}_1 + \vec{E}_1 \times \vec{B}_2 + \vec{E}_2 \times \vec{B}_1 + \vec{E}_2 \times \vec{B}_1 \right) dV , \tag{73} $$

and instead of Eq. (72), we would obtain

$$ \vec{F} = \frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = -q_1 \frac{d\vec{A}_{21}}{dt} - q_2 \frac{d\vec{A}_{12}}{dt} - \frac{d}{dt} \int_V \left(\vec{E}_1 \times \vec{B}_1 \right) dV - \frac{d}{dt} \int_V \left(\vec{E}_2 \times \vec{B}_2 \right) dV . \tag{74} $$

which does not agree with the law of conservation of the canonical momentum. Moreover, at the location of point-like charges the third and fourth integrals in *rhs* of Eq. (74) diverge. The difference between Eqs. (72) and (74) reflects a physical meaning of the “gauge renormalization”, when the time rates of the terms, taken from the same source particles ($\left(\vec{E}_1 \times \vec{B}_1\right)$ and $\left(\vec{E}_2 \times \vec{B}_2\right)$) contribute to their own EM momentum, associated with the EM mass, and thus represent the consequences of an action of the external forces, but not their cause.

5. Conclusions

1. In this paper we have removed the inconsistency that existed up to now in classical electrodynamics. Namely, in the gauge transformation of canonical energy-momentum tensor (1) to the symmetric form, we applied the non-homogeneous Maxwell equation (10) instead of the irrelevant homogeneous equation (6). As a result, the symmetric energy-momentum tensor acquired the form (16). This allows a gauge transformation, converting the divergent terms of classical electrodynamics to converging integrals. This operation was named as “gauge renormalization”.

2. Within the developed method of “gauge renormalization” we have got a full phenomenological description of the classical electron. We have found a limited value r_0 with the order of magnitude of the classical radius of electron; the electron’s charge density ρ falls off as $1/r$ inside the electron ($r \le r_0$), and $\rho = 0$ for $r > r_0$. We point out that this charge density becomes infinite at r=0. This singularity allows interpreting the classical electron as a point-like particle indeed, although the radius r_0 also has a certain physical meaning, as far as the charge density remains non-vanished at $r \le r_0$. The classical electron is not an unstructured particle. Otherwise, we inevitably get a variety of the problem “4/3”. A consistent description of the electron is carried out with its mechanical energy-momentum tensor, which includes the “Poincaré stresses”. We obtained this tensor in the form

$$T^{\mu\nu} = \frac{2q^2}{3r_0 V_e}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}.$$

This expression implies the following relationships for the mass components of the classical electron:

$$M = \frac{7q^2}{6r_0 c^2},\ M_{EM} = -M_P = \frac{2q^2}{3r_0 c^2}.$$

Due to the latter equality, the observed electron’s mass is purely mechanical. It is worth to mention that the equality $M_{EM} = -M_P$ does not mean corresponding equalities for the mass densities. In particular, the masses M and M_P are convicted inside the electron, whereas its electromagnetic mass M_{EM} is distributed over the entire space.

The developed approach fully resolves any problems, like “4/3 puzzling”, and provides the Lorentz-invariant description of the classical electron. In addition, we pay the attention to application of the Gauss theorem and the laws of electrostatics and magnetostatics at the scale less than the classical radius of electron r_0. It is known that an actual limit of applicability of classical electrodynamics has the order of 10^{-10} cm [5]. Hence it would be too naïve to prescribe the derived above properties of classical electron to real electron. Nevertheless, this paper demonstrates the self-consistence of classical electrodynamics in description of micro-charges, at least as their bound (non-radiative) electromagnetic fields are considered.

3. The obtained EMEM tensor (19) has been applied to derive the motional equation, the energy balance equation, and the momentum conservation law for the system of moving charged

particle. The motional equation for a non-radiating charged particle looks similar to the conventional one, but does not contain any self-forces. The motional equation for a radiating particle does not yield any "runaway solutions". The energy flux in a free EM field is guided by the Poynting vector. The energy flux in a bound EM field is described by the generalized Umov vector, defined in the paper.

4. According to classical electrodynamics after the "gauge renormalization", an electromagnetic momentum of a bound EM field for a system of charged particles consists of two parts: the electromagnetic momentum, associated with the EM masses of particles; the electromagnetic momentum, associated with the EM interaction of particles. The time rate of the first part of EM momentum is rather the consequence than the cause of the force experienced by the particle. Hence the forces, acting on charged particles, are determined by the time rate of the second (interaction) part of EM momentum.

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