A NON-EINSTEINIAN DEVELOPMENT OF THE PCC RELATIVITY THEORY

Lyubomir T. Gruyitch
University of Technology Belfort - Montbéliard
90010 Belfort - Cedex, France
lyubomir.gruyitch@utbm.fr

The paper is devoted to the memory of the great scientist and wonderful person, late Professor Dr

Paul MARMET
(May 20th, 1932, Québec City, Québec, Canada - May 20th 2005, Ottawa, Ontario, Canada)

Abstract.

The goal is to present a non-Einsteinian (a non-Lorentz-Einstein-Poincaré) development of the Partially Compatible but Consistent (PCC) relativity theory, which is based on the inherent characteristics of time, to show its consequences in the fundamentals of physics and influences on the further development of the relativity theories.

It was explained in the former papers that time is the unique physical variable, which agrees with Newton's explication of the absolute sense of time. Its value has been changing continuously monotonously increasingly equally and uniformly in all directions independently of anybody and anything. This should be reflected in the transformations of the temporal co-ordinates, which is a priori rejected in the Einsteinian relativity theory. The time scaling coefficients were a priori accepted equal in the Lorentz transformations, hence in the Einsteinian relativity theory, by restricting its validity only to such cases. In order to express the time independence, all the scaling coefficients will be permitted a priori different.

Another characteristics of Einstein's approach, which characterises the Einsteinian relativity theory, is the a priori acceptance of a light signal for the reference spatial point, hence of the light speed for the reference spatial speed, and the determination of the scaling coefficients exclusively for such a case. This implied partial pairwise compatibility of the transformations and represents another restriction on the validity of the Einsteinian relativity theory. In order to weaken this restriction, an arbitrary point with an arbitrary speed will be the spatial reference point and the spatial reference speed.

Einstein accepted the assumption on the invariance of the light speed relative to inertial frames (and tacitly relative to all time axes). He lifted it to the level of the uncontested postulate that has been the keystone of the Einsteinian relativity theory, which has also caused that the Einsteinian relativity theory is a singular case among relativity theories. The non-Einsteinian approach of this paper relaxes the theory of such a restrictive assumption. The values of all quantities will be consistently expressed with respect to the corresponding units.

The general case of the co-ordinate transformations is free of all constraints a priori accepted in the Einsteinian relativity theory. Consequently, new formulae are substantially different from those in the Einsteinian relativity theory. In the special and the singular case the formulae become of the form known in the Einsteinian relativity theory, but still more general. They reduce to the results of the Einsteinian relativity theory in the particular singular case.

A result of the paper is that for every speed, and not only for the light speed, we can find co-ordinate transformations such that the speed is invariant relative to such transformations. Such transformations for the light speed are the Lorentz transformations. This illustrates that the Lorentz-Einstein invariance of the light speed is not the property of light, hence not of its speed.

The paper opens a new avenue for research in physics, in the relativity theory, in mathematics and in the theory of dynamical systems with multiple time scales.

Keywords: Basic physics, co-ordinate transformations, dynamical systems Lorentz transformations mathematical physics, relativity theory, systems with multiple time scales, time.
1. INTRODUCTION

The basic assumption of Einstein's relativity theory, which Einstein lifted to the unquestionable postulate, is the invariance of the light speed. Another keystone of his theory is time dependence of the frames, hence of the space. He equalised time (the temporal variable) with its value. He did not distinguish between variable (time) and its value (time value). Consequently, he treated relativity of the numerical time value, of time unit and of time scale as the relativity of time itself. [4] – [6].

Lorentz, Einstein and Poincaré used the Voigt transformations in the form known as the Lorentz transformations in which the spatial transfer speed is invariant relative to all time axes (relative to zero instant, initial instant, time unit, and time scale). They a priori accepted also a light signal to be the reference temporal and spatial point, and its speed to be the reference and generic speed. [4] - [6], [29] – [34], [49].

The a priori accepted assumptions of Einstein's relativity theory constrain its validity so sharply that it represents a singular case, rather than to hold in general [17] - [25].

If we consider Einstein's explanation of time relativity only in the sense of the relativity of a zero time value, of an initial time value, of a time unit, and of a time scale, then it is in Newton's sense (Scholium I on the page 8 in [48]). However, if we interpret Einstein's explanation of time relativity as the relativity of the variable - time, then it opposes crucially Newton's explication of time and of its relativity, which are essentially correct [13], [14], [17] – [25].

Time does not force any clock to function. Time does not depend on the clock speed. It is a kind of energy (exchange) that forces the clock to work. If the clock is movable, then it is a kind of energy, not time, which causes its movement. What clock hands show, it is not time, but it is a number associated with the time value relative to the chosen initial time value, time scale and time unit. A change of the speed of the clock hand, whatever is its cause, induces automatically the corresponding change of the time unit, but not any variation of time itself. Various speeds of various clock hands correspond to various time scales and time units. They do not, and cannot represent various time speeds. A change of the speed of a clock hand does not express a change of the time speed. Marmet presented excellent energy - matter explanations of these facts, [35], and discovered other serious drawbacks of the Einsteinian relativity theory (which is the Lorentz - Einstein - Poincaré theory of time relativity and its developments), [35] - [46]. Other physical explanations, mathematical proofs and more details are given in [25].

The recent papers [22] - [24] explained the importance of compatibility of the co-ordinate transformations. More details are in [25]. They present the definitions of partial and complete, either pairwise or entire, compatibility of the transformations. The Lorentz transformations are restrictively pairwise compatible rather than completely pairwise compatible, [22] - [25]. Besides, these references explain inconsistency of the use of the values of speeds in the Lorentz transformations and in, from them deduced, velocity and acceleration transformations.

This opened new problems in the relativity theory - the problem of complete (pairwise, entire) compatibility of the transformations, the problem of the consistent use of values of all variables in the transformations, and raised the following questions:

Q1 What are the necessary and sufficient conditions for the temporal co-ordinate transformations and for the spatial co-ordinate transformations to be (partially or completely, pairwise and/or entire) compatible?

Q2 What are the necessary and sufficient conditions for the velocity transformations to be (partially or completely) compatible?

Q3 What are the necessary and sufficient conditions for the acceleration transformations to be (partially or completely) compatible?

The new results obtained in [22] - [25] provoke also the following questions:

Q4 Does the constancy of the light speed in vacuum means also its invariance?

Q5 What is the justification for the invariance of the spatial transfer speed value in the Lorentz transformations?

Q7 What are consequences of the consistent application of the values of all variables relative to the frames used in the co-ordinate, velocity and acceleration transformations?

Q7 What are implications of different time scales on the co-ordinate, velocity and acceleration transformations, hence, on the relativity theory?

Q4 What are implications on dynamical systems with multiple time scales?

Q5 What are implications on speed, mass and energy?

The papers [22] - [24] contribute with full replies to these questions in the framework of the uniform...
transformation, and the book [25] replies to the questions regardless of non-uniformity or uniformity of the transformations. Uniformity of the transformations expresses their property that the same initial moment, the same time scale and time unit, i.e. the same time set and time axis, hold over the whole space. This is the temporal uniformity over the space. Another kind of the uniformity is the spatial uniformity. It is achieved in the Einsteinian relativity theory by accepting a priori a light signal for the spatial reference point.

The new co-ordinate, velocity and acceleration transformations in [22] - [25] represent either Newtonian generalisations of the Lorentz transformations or novel transformations essentially different from the Lorentz transformations and, from them deduced, velocity and acceleration transformations. They are based on the properties of time, which we discover in the physical reality.

The original form of the basic postulate of Einstein’s relativity theory is Einstein's principle of the constancy of the light speed in vacuum relative to inertial frames. This is surprisingly interpreted and used after Einstein himself as the uncontested postulate on the invariance of the (numerical) value of the light speed with respect to a choice of an inertial spatial frame and of a time axis. Surprisingly, because the numerical value of the relative speed of anybody or of anything depends on both the speed of the inertial frame and on the units of both time and space. The value of the light speed in vacuum relative to inertial frames. Their values are determined exclusively for a priori accepted speed of an arbitrary point to be the light speed [4] – [8], [29] – [34], [49].

The aim of this paper is to present new co-ordinate and velocity transformations with the consistent use of the numerical values of all variables. They will be relaxed of the a priori accepted restrictions of the Einsteinian relativity theory. The reference spatial point will be freely chosen and fixed, rather than to be a light signal that is a priori accepted in the Einsteinian relativity theory. Consequently, the new transformations are partially compatible and consistent. They belong to the new mathematical relativity theory called Partially Compatible but Consistent (for short: PCC) Relativity Theory [25]. It overcomes the drawbacks of the Einsteinian relativity theory by new solutions to the posed problems.

2. Notation

We may accept any time scale for a time axis \( T \). Different time scales can be associated with different time axes: “an original time scale” \( T \) that is not indexed, and “\( k \)”-time scale \( T_k \). A time value (instant, moment) measured in \( T \)-scale and in \( T_k \)-scale is designated by \( t \) and \( t_k \), respectively, \( t \in T \) and \( t_k \in T_k \), \( k = 1, 2, \ldots, s \). The indices \( i, j \) are arbitrary and fixed, \( i, j \in \{ 1, 2, \ldots, s \} \). They may be equal, \( i = j \), or different and then \( i < j \). If they are equal, which is the trivial case, the equations become identities.

The Cartesian product set \( T \times R^n \) is denoted by \( I = T \times R^n \). It is the \((1+n)\)-dimensional real integral vector space, for short the integral space. A pair \((t, x)\) is called an event in \( I \) [24] through [26]. It can occur exactly once due to the properties of time. The integral space corresponding to the time axis \( T_k \) and to the frame \( R_k \) is \( I_k: I_k = T_k \times R^m_k \), \( k = 1, 2, \ldots, s \).

The vector \( u \) is an arbitrarily chosen and fixed constant unity vector \( \| u \| = 1 \), where \( \| . \| \) is the Euclidean (or any other scalar) norm. It is elementwise different from the zero vector.

Let origin \( O_k \) of the co-ordinate system \( R_k \), hence \( R_k \) itself, move with a constant velocity \( v_{Ok} \) relative to the origin \( O \) of \( R^n \) and measured with respect to \( t \in T \). The norm \( \| . \| \) of the velocity \( v_{Ok} = v_{Ok} u \) is denoted by \( v_{Ok} \). \( v_{Ok} = \| v_{Ok} \| \) if measured relative to \( t \in T_k \), \( k = 1, 2, \ldots, s \). It is the speed of the origin \( O_k \) relative to the origin \( O \) of \( R^n \).

The value of the light speed in vacuum is denoted as usually by \( c \) [5: p. 15], [6: p. 26]. It is the light speed value with respect to a stationary frame in vacuum and measured relative to \( I \), for short: the light speed value.
We can accept to consider a light signal and a translation of $O_i$ together with $R^n_i$ in a direction and sense of the unity vector $u$. The constant unity vector $u$ is used to represent symbolically also the direction and the sense of a movement of the space $R^n_i$. Let $\text{sign}(x)=|x|^{-1}x$ for $x \neq 0$, and sign $0 = 0$. Without losing in generality, [25], we represent a position of an arbitrary point $P$ relative to the origins $O_i$ and $O_j$ of $R^n_i$ and $R^n_j$ at the same moment by vectors $r_P \in R^n_i$ and $r^j_P \in R^n_j$ as $r_P = r_{P_0}u$, and $r^j_P = r^j_{P_0}u$, $i = 1, 2, \ldots, s$, respectively. Their lengths and senses are expressed by their norms $r_P = |r_P|$ and $r^j_P = |r^j_P|$, and by their algebraic values $r_P = \rho P \text{sign}(u^T r_P)$ and $r^j_P = \rho^j P \text{sign}(u^T r_P)$, respectively, which may vary in time:

$$r_P(t_{ij}; t_{i0}) = r_P(t_{ij}; t_{i0})u \quad \text{is the position vector of the point } P \text{ with respect to } O_{ij} \text{ at } t_{ij}, (.) = i, j.$$ 

Their velocities depend on the reference integral space in general:

$c^j_i$ is the scalar value of the light speed measured with respect to the origin $O_i$ of $R^n_j$ and relative to $t_i$ (rather than relative to $t_j$ if $T_i \neq T_j$),

c^j_i$ denotes both $c^j_i$ and $c^i_j$ if and only if $c^j_i = c^i_j$;

c^j_i = c^j_i = c^j_i$, which is denoted simply by $c$ in the Lorentz transformations (L1) through (L4) because it has been considered in the Einsteinian relativity theory as invariant with respect to a choice of the integral space,

$c^j_i$ is the light speed relative to the integral space

$I_{ij} = T_{ij} x R^n_{ij}, (.) = i, j; i, j \in \{1, 2, \ldots, s\}, i \leq j,$

$P$ an arbitrarily chosen point in the space $R^n_i$, $P_R$ a freely chosen and then fixed temporal reference point,

$P_{SU}$ a freely chosen and then fixed spatial reference point in the space $R^n_i$,

$v^0_P(t_{ij}; t_{i0}) = v^0_P(t_{ij}; t_{i0})u$ is the instantaneous velocity (the speed vector) of the point $P$ with respect to $O_i$ at $t_i$ and measured relative to $t_{ij}$ provided the initial moment was $t_{i0}$;

$v^0_P(t_{ij}; t_{i0}) = v^0_P(t_{ij}; t_{i0})u$ if and only if the velocity $v^0_P(t_{ij}; t_{i0})$ is constant vector, and

$v^0_P(t_{ij}; t_{i0}) = v^0_P(t_{ij}; t_{i0})u$ if and only if $t_{i0}$ is known and fixed,

$v^0 m_{O_k} = v^0 m_{O_k}u$ is the constant velocity of $O_k$ relative to $O$ measured in terms of $t_m$, $k, m \in \{1, 2, \ldots, s\},$

$v^0_{O_j} \leq v^0_{O_j}$ is accepted, $r \in \{i, j\},$

$v^k_{ij} = v^k_{ij} u = (v^0_{O_j} - v^0_{O_i})u$ is the constant relative velocity of $O_j$ with respect to $O_i$ measured all in terms of $t_k, k \in \{i, j\}$; notice that $v^j_i = -v^i_j$, that

$v^j_i \neq v^j_i = (v^j_{O_j} - v^j_{O_i})u = -v^j_i$ is possible if $T_{ij} \neq T_{ij}$, \text{ that } v^{ij}_{ij} = v^{ij}_{ij}u \text{ is called the spatial transfer velocity and that } v^{ij}_{ij} \text{ is called the spatial transfer speed,}$

$v^{ij}_{ij} = v^{ij}_{ij} u$ denotes both $v^{ij}_{ij}$ and $v^{ij}_{ij}$ if and only if they are equal: $v^{ij}_{ij} = v^{ij}_{ij} = -v^{ij}_{ij}$. In the Lorentz transformations there is $v^{ij}_{ij}$ denoted by $v$ because it is considered invariant. If, and only if $i = j$ then $v^{ij}_{ij} = -v^{ij}_{ij} = v^{ij}_{ij} = -v^{ij}_{ij} = v = 0$,

$v^{ij}_{SU}$ is the known constant speed of the spatial reference point $P_{SU}$; $v^{ij}_{SU} \in R^n$; $v^{ij}_{SU} \in \{c^{ij}_{ij}, v^{ij}_{SU}\}$ is permitted but not required,

$q^{ij}, \omega^{ij}$ are the generic speeds, the values of which are measured relative to $T_{ij}$, and which obey the following:

$q^{ij}, \omega^{ij} \in R^n$, and we permit

$$q^{ij}, \omega^{ij} \in \left[\left[c^{ij}_{ij}, v^{ij}_{SU}\right] \left[v^{ij}_{SU}, v^{ij}_{SU}\right]\right], (.) = i, j; i, j \in \{1, 2, \ldots, s\}, i \leq j.$$

If only if the point $P$ represents a light signal L, then:

$r^0_P = r_L = r \in R^n, r^0_L = r_L u, r^{ij}_{SU} = r^{ij}_{SU} \in R^n, \text{ and}$

$v^{ij}_{SU} = v^{ij}_{SU} u = v^{ij}_{SU} = v^{ij}_{SU} u$,

$(.) = i, j; i, j \in \{1, 2, \ldots, s\}, i \leq j.$

3. \textit{Time Features}

When we speak about relativity then we think of the relativity of \textit{time}. What is \textit{time}? Is it that what the clock hands show as Einstein claimed? What are its features? We cannot speak exactly about its relativity before we reply to these questions.

Our knowledge of, the common sense based on, and the experience with the physical reality led to the following axiomatic form of the characterisation of \textit{time}, which expresses its crucial properties, agrees with Newton's explanation of \textit{time} [48], and which has
permitted new fundamental results in the relativity theory [13], [14], [17] – [25]:

**Axiom 1.**

a) **Time**

*Time (temporal variable)* denoted by \( t \) (or by \( \tau \)) is an independent physical variable. Its value occupies (covers, encloses, impregnates, is over and in, and penetrates) equally, always and everywhere, beings, energy, matter, objects, and the space. Its value has been and will be smoothly and strictly monotonously continuously increasing always equally in all directions in space and independently of beings, energy, matter, objects and space, independently of all other (e.g. physical and mathematical) variables, independently of all movements, of all processes, of all events and of the space.

b) **Physical dimensionality of time**

Time is its own unique component. It is a basic physical variable. Its physical dimensionality is a basic physical dimensionality denoted by “\( T \)”, \( t \in [T] \), where “\( T \)” means “time”.

c) **Time value and its flow**

The value of time is called moment or instant. It is the shortest possible duration that is the duration of a single time value. It can happen exactly once and then it is the same for all beings, energy, matter, for all objects, in the whole space, for all events, movements and processes. It is denoted by \( t \) and a subscript, e.g. \( t_a \). An arbitrary instant will be denoted as time itself by \( t \). The time value is determined accurately up to an unknown additive constant. A sequence of time values determines uniquely the order of events happening.

A flow (i. e. an oriented variation) of time values (for short: a temporal flow) from one time value to another one is the duration between them (for short: duration). Its orientation is the temporal orientation (the temporal sense, the temporal arrow), which is from a smaller (from a past) time value to a bigger (to a future) time value: \( dt > 0 \).

There can be assigned exactly one real number to every moment, and vice versa. The set \( T \) of all moments is in one-to-one correspondence with the set \( R \) of all real numbers. An accepted rule of the correspondence determines a relative zero numerical value of time, a time scale and a time unit, and vice versa.

A total zero moment \( t_{0\text{Total}} \) has not existed and will not occur. Any instant may be accepted for a relative zero time value \( t_{0} \).

d) **Time interval**

The temporally ordered connected set of all instants between two different time values is a time interval. The time interval \( [t_0, t_{\text{trm}}] \) reflects the duration from the initial instant \( t_0 \) to the terminal instant \( t_{\text{trm}} \) either of the existence or of the non-existence of the related being, or of the related form of energy, of the related kind of matter, of the related object, of the related event, of the related movement, of the related process or of the related rest.

e) **Age**

*Moment (instant)* reflects an instantaneous temporal situation of somebody or of something, which is called her/his or its age. Time value difference \( t – t_0, t > t_0 \) (or equivalently, time interval \([t_0, t] \) expresses, and it is used to measure the duration of the (non-) existence of a being, of a form of energy, of a kind of matter, of an object, of a movement, of a process, of a rest; it is used to measure duration of the (non-) existence of somebody or of something, relative to an accepted initial moment \( t_0 \). If \( t_0 \) is the instant of the beginning of the (non-)existence of a being, of a form of energy, of a kind of matter, of an object, of a movement, of a process, of a rest, of somebody or of something, then the time value difference \( t – t_0, t > t_0 \), (or equivalently, the time interval \([t_0, t] \) represents the age of the being, of the form of energy, of the kind of matter, of the object, of the movement, of the process, of the rest, of the somebody or of the something.

A co-ordinate axis used for a time axis is in the sequel immovable relative to the environment. It is denoted by \( T \). It is a geometrical representation of the time set denoted also by \( T \), which is the temporally ordered set of all instants:

\[
T = \{ t; t \in R, t \in C^1(R), dt > 0 \}.
\]

Evidently, \( T \neq R \) due to \( dt > 0 \).

Once a time axis has been accepted with a fixed time scale including a fixed time unit, then the (relative) zero moment \( t_{0} \) has been tolerated. Afterwards, we can select any instant \( t_0 \in T \) for an initial instant. We accept that it has been chosen, known and fixed. It can be \( t_0 = 0 \), but need not.

The complete characterisation of time and its detailed explanation of its properties are presented in [25].

We should recognise the fact, which has been often ignored, that Newton himself [48: pp. 8 – 10] introduced and explained both the absolute and relative sense of time. Moreover, we should appreciate the truth
that Newton’s explanation of the relativity of time incorporates that of Einstein [4, p. 20], [6, pp. 26 – 27], [7, pp. 23 – 40], which was explained and proved in [13], [14], [17] – [25].

4. DYNAMICAL SYSTEMS WITH MULTIPLE TIME SCALES

Different processes can propagate with different speeds in different beings or systems, which can hold also in the same being or system at its different points or parts. Therefore, different time scales, different initial moments and/or different time units can be assigned to different parts of the same material object giving a relative meaning to time (in this sense that is Newtonian). This was recognised and used in the theory of dynamical systems with multiple time scales, which include singularly perturbed systems.

A large class of dynamical systems is adequately mathematically modelled by (S1) and (S2), [10] – [12], [15].

\[
\frac{dx}{dt} = f_1(t, x, y, M), x \in R^P, y \in R^S, \quad (S1)
\]

\[
f_1(.) : R^P \times R^S \times R^{xS} \to R^P, M \in R^{xSx},
\]

\[
M \frac{dy}{dt} = f_2(t, x, y, M), \quad (S2)
\]

The matrix \( M = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_s\} \) contains different (possibly, but not necessarily, small) parameters \( \mu_k \) that enable the introduction of \( s \) different time scales and/or units, \( \mu_k \in R^+ \), \( k = 1, 2, \ldots, s \), \( R^+ = \{x : t_k - t_0 = \mu_k(t - t_0), t_0 = \mu_k \in R^+ \} \).

The link between the theory of dynamical systems with multiple time scales and the relativity theory is crucial to overcome the drawbacks and the constraints of the Einsteinian relativity theory. It enables the system approach to the relativity theory [13], [14], [17] - [25]. The time scaling coefficients \( \mu_k \) establish the link.

5. COMPATIBILITY OF THE TRANSFORMATIONS

The time scaling coefficients \( \alpha^j_i \) and \( \alpha^j_i \), and the space scaling coefficients \( \lambda^j_i \) and \( \lambda^j_i \) should be positive real valued. They intervene in the following generic co-coordinate transformations (2) through (5):

\[
t_1 - t_{i0} = \alpha^j_i(t_j - t_{j0}) + \frac{v^j_i}{q^i j} r_j(t_j - t_{j0}),
\]

\[
i, j \in \{, 1, 2, \ldots, s\}, i \leq j
\]

\[
t_j - t_{j0} = \alpha^j_i(t_i - t_{i0}) - \frac{v^j_i}{q^i o} r_i(t_i - t_{i0}),
\]

\[
i, j \in \{, 1, 2, \ldots, s\}, i \leq j
\]

\[
r_j(t_j; t_{j0}) = \lambda^j_i[r_j(t_i; t_{i0}) + v^j_i(t_j - t_{j0})],
\]

\[
i, j \in \{, 1, 2, \ldots, s\}, i \leq j
\]

\[
r_j(t_j; t_{j0}) = \lambda^j_i[r_j(t_i; t_{i0}) - v^j_i(t_i - t_{i0})],
\]

\[
i, j \in \{, 1, 2, \ldots, s\}, i \leq j
\]

The time scaling coefficients \( \alpha^j_i \) and \( \alpha^j_i \), and the space scaling coefficients \( \lambda^j_i \) and \( \lambda^j_i \) are a priori mutually different, which expresses time independence of the space.

The co-ordinate transformations (2) through (4) do not contain the value of the speed of light. In a special case \( q^{(i_0)} = 1 \) that characterises the Lorentz transformations. Then the co-ordinate transformations contain the value of the speed of light.

The condition for the preservation of the generalised length in integral spaces will be used in its general form (6).

\[
\left[ \begin{array}{c}
q_{O}^1 P_i \\
(t_1 - t_{i0})v_{P_i}^{O_i}(t_i)
\end{array} \right]^T \begin{bmatrix}
G \quad 0 \\
0 \quad G
\end{bmatrix}
\left[ \begin{array}{c}
q_{O}^1 P_i \\
(t_1 - t_{i0})v_{P_i}^{O_i}(t_i)
\end{array} \right] =
\]

\[
\left[ \begin{array}{c}
r_{P_i}^{O_i} \\
(t_1 - t_{i0})v_{P_i}^{O_i}(t_i)
\end{array} \right]^T \begin{bmatrix}
G \quad 0 \\
0 \quad G
\end{bmatrix}
\left[ \begin{array}{c}
r_{P_i}^{O_i} \\
(t_1 - t_{i0})v_{P_i}^{O_i}(t_i)
\end{array} \right] .
\]

The matrix \( G = \text{blockdiag}\{A, -B\} \) in (6), and the matrices \( A \) and \( B \) are positive definite matrices with \( A = B \) possible but not required, \( A \in R^{nxn} \). The block diagonal form of the matrix \( G \) reflects the time independence of the space (Axiom 1).

The temporal co-ordinate equations (1) are inherent for multiple time scale dynamical systems [13], [14], [17], [20]. The condition (6) expresses the Gaussian transformation of the time-space length. It is a crucial condition of Einstein’s general relativity theory for validity of the co-ordinate transformations defined by (2) through (5), [4], [5].

If, and only if the transformations (1) through (5) obey (6) then they form the *Poincaré group*. 

The generic co-ordinate transformations (2) - (5) can be set into the form of the homogeneous linear equations with variable gains [25] in general. The same holds for the Lorentz transformations.

**Definition 1.** [24], [25]

a) The temporal co-ordinate transformations (2) and (3) are **compatible** if, and only if they yield an identity as soon as one temporal co-ordinate and the corresponding (with the same subscript) spatial co-ordinate are eliminated from them [without using the temporal co-ordinate transformations (4) and (5)]. Otherwise, they are **incompatible**.

Likewise, the spatial co-ordinate transformations (4) and (5) are **compatible** if, and only if they yield an identity as soon as one spatial co-ordinate and the corresponding (with the same subscript) temporal co-ordinate are eliminated from them [without using the temporal co-ordinate transformations (2) and (3)]. Otherwise, they are **incompatible**.

b) The transformations (2) through (5) are **pairwise compatible** if, and only if both the temporal co-ordinate transformations (2) and (3) are compatible and the spatial co-ordinate transformations (4) and (5) are compatible. Otherwise, they are **pairwise incompatible**.

c) The transformations (2) through (5) are **entirely compatible** if, and only if both 1) and 2) hold:

1) The temporal co-ordinate transformations (2) and (3) yield, by means of the spatial co-ordinate transformations (4) and (5), an identity as soon as temporal and spatial co-ordinates with the same subscripts are eliminated from them.

2) The spatial co-ordinate transformations (4) and (5) yield, by means of the temporal co-ordinate transformations (2) and (3), an identity as soon as temporal and spatial co-ordinates with the same subscripts are eliminated from them.

Otherwise they are **entirely incompatible**.

d) The transformations (2) through (5) are **partially (i.e. restrictively) pairwise, entirely compatible** if, and only if they are, respectively, pairwise, entirely compatible exclusively when the arbitrary point P moves with the speed restricted by certain conditions (which do not allow its arbitrary non-zero value), e. g. with the light speed, or when the generic speeds \( \omega^{(i)} \) and \( \omega^{(j)} \) should obey certain constraints.

e) The transformations (2) through (5) are **completely (pairwise, entirely) compatible** if, and only if they are, respectively, (pairwise, entirely) compatible regardless of the speed of the arbitrary point P different from the light speed and if there is not any constraint on \( \omega^{(i)} \) and \( \omega^{(j)} \).

**Definition 2.** [24], [25]

The co-ordinate transformations (2) through (5) are **consistent** if, and only if the same time unit, the same length scale, and the same length unit are applied to the values of all the variables related to the same integral space.

6. **CONSTRAINTS OF THE EINSTEINIAN RELATIVITY THEORY**

The original Lorentz transformations were determined under the following a priori accepted conditions, which were also a priori adopted by Einstein, hence in the Einsteinian relativity theory [3] - [8], [29] - [34], [49]:

**Constraint 1.** All the time scaling coefficients \( \alpha^{(i)}_j \) and \( \alpha^{(j)}_i \) are equal: \( \alpha^{(i)}_j = \alpha^{(j)}_i = \alpha \).

**Constraint 2.** All the space scaling coefficients \( \lambda^{(i)}_j \) and \( \lambda^{(j)}_i \) are equal: \( \lambda^{(i)}_j = \lambda^{(j)}_i = \lambda \).

**Constraint 3.** In the proofs of the Lorentz transformations it is accepted that the arbitrary point P moves exclusively with the velocity of light:

\[
\mathbf{v}^{(i)}_P(t_j) = c^{(i)}_u(T, j), i, j \in \{-1, 2, \ldots, s\},
\]

**Constraint 4.** The numerical value of every speed, including the value \( c^{(i)}_u \) of the light speed, is invariant relative to time axes:

\[
\mathbf{v}^{(i)}_P(T_j) = \mathbf{v}^{(i)}_P(T_j), \quad c^{(i)}_u(T, j), i, j \in \{-1, 2, \ldots, s\}, i \leq j.
\]

and the value of the light speed, hence its numerical value as well, is also invariant with respect to inertial spatial co-ordinate systems:

\[
c^{(i)}_u(T, j) = i, j \in \{-1, 2, \ldots, s\}, i \leq j.
\]

**Constraint 5.** In the proofs of the Lorentz transformations it is accepted that the position of the
arbitrary point $P$ is the position of a light signal: $r_p(t_{ij};t_{ij0}) = r_L(t_{ij};t_{ij0}) = c(t_{ij} - t_{ij0}u)$, $r_L(t_{ij};t_{ij0}) = r_L(t_{ij};t_{ij0}) \in \mathbb{R}_+$, $\forall t_{ij} \in T_i$, $(.) = i,j; i,j \in \{-1, 2, \ldots, s\}, i \leq j$.

**Constraint 6.** The matrices $A$ and $B$ in $G$, (6), are equal, $A = B \rightarrow G = \text{blockdiag}\{A - A\}$.

In Einstein’s special relativity they are the identity matrix $I$, $A = B = I$.

These constraints will be omitted a priori in the sequel.

7. **Case of the partially compatible but consistent (PCC) relativity theory**

7.1. **Problem Statement**

We will distinguish two essentially different cases with respect to the mutual relationship among the scaling coefficients, which are constant in either case:

- **General case:** $\alpha_i^j \neq \alpha_i^j$ and/or $\lambda_i^j \neq \lambda_i^j$.
- **Special case:** $\alpha_i^j = \alpha_i^j = \alpha_{ji} = \alpha_{ij}$ and $\lambda_i^j = \lambda_i^j = \lambda_{ji} = \lambda_{ij}$.

A particular sub-case of the special case concerns the invariance of the light speed and of the transfer speed relative to a choice of a time axis. Besides, $\alpha_{ji} = \alpha_{ij} = \alpha$ and $\lambda_{ji} = \lambda_{ij} = \lambda$. It is the singular case.

**Problem statement.** Determine the values of the scaling coefficients $\alpha_i^j, \alpha_i^j, \lambda_i^j, \lambda_i^j, \text{and } \mu_i, (1) - (5)$, for the movement of the arbitrary point $P$ with an arbitrary velocity, and for an arbitrarily chosen and then fixed spatial reference point $P_{SU}$ moving with a known constant velocity $v_{SU}^{(i)} = v_{SU}^{(i)}u$, so that they are positive real numbers, that the equations (2) through (5) hold, and that they together with (1) imply the identity (6). Verify compatibility of the resulting co-ordinate transformations.

7.2. **Problem solutions for the general case**

We will allow for the (numerical) value of any speed, hence of the light speed, to depend on an integral space with respect to which it is determined.

**Theorem 1.** Let the time scaling coefficients $\mu_i$ be defined by (1). Let the spatial reference point $P_{SU}$ be arbitrarily chosen and then fixed moving with a known constant velocity $v_{SU}^{(i)} = v_{SU}^{(i)}u, v_{SU}^{(i)} \in \mathbb{R}^+$. If the speed of the arbitrary point $P$ is arbitrary, then, in order for the scaling coefficients $\alpha_i^j \in \mathbb{R}^+, \alpha_i^j \in \mathbb{R}^+, \alpha_i^j \neq \alpha_i^j, \lambda_i^j \in \mathbb{R}^+$ and $\lambda_i^j \in \mathbb{R}^+, \lambda_i^j \neq \lambda_i^j$, determined for the speed $v_{SU}^{(i)}$ of $P_{SU}$, to be constant and to obey the equations (2) through (5), and for (1) through (5) to imply (6), it is necessary and sufficient that the following equations hold for any choice of the time scaling coefficients $\mu_i \in \mathbb{R}^+$ and $\mu_j \in \mathbb{R}^+$:

\[
\begin{align*}
\alpha_i^j &= \mu_i \frac{1}{\mu_j} \frac{v_{SU}^{(i)}}{v_{SU}^{(i)} + v_{SU}^{(i)}u} = \frac{v_{SU}^{(i)}}{1 + \frac{v_{SU}^{(i)}u}{q^2 \omega^2}}, \\
i_j \in \{-1, 2, \ldots, s\}, & i \leq j
\end{align*}
\]

(7)

\[
\begin{align*}
\alpha_i^j &= \mu_i \frac{1}{\mu_j} \frac{v_{SU}^{(i)}}{1 - \frac{v_{SU}^{(i)}u}{q^2 \omega^2}} = 1, \\
i_j \in \{-1, 2, \ldots, s\}, & i \leq j
\end{align*}
\]

(8)

\[
\lambda_i^j = \frac{1}{v_{SU}^{(i)}}, i_j \in \{-1, 2, \ldots, s\}, i \leq j, \quad \lambda_i^j = \frac{1}{v_{SU}^{(i)}}, i_j \in \{-1, 2, \ldots, s\}, i \geq j
\]

(9) (10)

\[
0 \leq v_{ji}^{(i)} < \min \left\{v_{SU}^{(i)}, \frac{q^2 \omega^2}{v_{SU}^{(i)}}, \lambda_{ij} \right\}, (.) \in \{ij\},
\]

(11)

\[
\begin{align*}
\mu_i = \frac{v_{SU}^{(i)}}{v_{SU}^{(i)}}, & c_i^j = c_i^j, c_i^j = \frac{v_{SU}^{(i)}}{v_{SU}^{(i)}}, i_j \in \{-1, 2, \ldots, s\}, i \leq j, \\
\mu_i = \frac{v_{SU}^{(i)}}{v_{SU}^{(i)}}, & c_i^j = c_i^j, c_i^j = \frac{v_{SU}^{(i)}}{v_{SU}^{(i)}}, i_j \in \{-1, 2, \ldots, s\}, i \geq j
\end{align*}
\]

(12)

The equations (7) – (10) give the next form to the equations (2) – (5):

\[
\begin{align*}
t_i - t_{i0} &= \mu_i \frac{(t_j - t_{j0}) + v_{ji}^{(i)}r_{P_{SU}}(t_j; t_{j0})}{1 + v_{ji}^{(i)}v_{SU}^{(i)}q^2 \omega^2}, \\
t_j - t_{j0} &= \mu_j \frac{(t_i - t_{i0}) - v_{ji}^{(i)}r_{P_{SU}}(t_i; t_{i0})}{1 - v_{ji}^{(i)}v_{SU}^{(i)}q^2 \omega^2}
\end{align*}
\]

(13) (14)
The transformations (13) through (16) are partially both entirely and pairwise compatible.

**Proof. Necessity.** Let the spatial reference point \( P_{SU} \) be freely chosen and fixed. Let it move with a known constant speed \( v_{SU}^\prime \in R^+ \). Let the arbitrary point \( P \) be \( P_{SU}, v_P^\prime = v_{SU}^\prime \). Let the time scaling coefficients \( \mu_i \in R^+ \), \( i = 1, 2, \ldots, s \), be defined by (1). Let the coefficients \( \alpha_i^j \in R^+\), \( \alpha_i^j \in R^+\), \( \alpha_i^j \neq \alpha_i^j \), \( \lambda_i^j \in R^+\), and \( \lambda_i^j \in R^+\), \( \lambda_i^j \neq \lambda_i^j \), be constant and obey (2) through (5), and let (1) through (5) imply (6). Since \( r_P(t_j; t_{i0}) = \equiv r_P(t_j; t_{i0}) \) represents the position of the arbitrary point \( P = P_{SU} \) measured relative to the integral space \( I_j = T_j \times R^n \), then \( r_P(t_j; t_{i0}) = O_{SU}^j(t_j - t_{i0}) \), \( j = -1, 1, 2, \ldots, s \). This, (1) and (2) yield:

\[
t_{i} - t_{i0} = \mu_i(t - t_{0}) = \alpha_i^j \left( 1 + \frac{v_{j}^\prime v_{SU}^\prime}{q^\prime \omega^\prime} \right) (t_j - t_{i0}) = \equiv \alpha_i^j \left( 1 + \frac{v_{j}^\prime v_{SU}^\prime}{q^\prime \omega^\prime} \right) \mu_j (t - t_{0}),
\]

so that:

\[
\alpha_i^j = \frac{\mu_j}{\mu_i} \left[ 1 + \frac{v_{j}^\prime v_{SU}^\prime}{q^\prime \omega^\prime} \right]^{-1}.
\]

This implies the first equation in (7). The first equation in (8) is analogously proved by combining \( r_P(t_1; t_{i0}) = v_{SU}^\prime(t_1 - t_{i0}) \) with (1) and (3). From (1) and from (4) we deduce the following:

\[
r_P(t_1; t_{i0}) = v_{SU}^\prime O_{SU}^i \mu_i(t - t_{0}) = \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) u,
\]

\[
\lambda_i^j = \frac{\mu_i v_{SU}^\prime}{\mu_j v_{SU}^\prime} \left( 1 + \frac{v_{j}^\prime}{v_{SU}^\prime} \right) \mu_j \left[ 1 + \frac{v_{j}^\prime v_{SU}^\prime}{q^\prime \omega^\prime} \right]^{-1}.
\]

We transform the right hand side of (6) as follows by using (1) and \( r_P(t_k; t_{i0}) = v_{SU}^\prime (t_k - t_{i0}) \), \( k = 1, 2, \ldots, s \):

\[
\left[ r_P(t_1; t_{i0}) \right]^T G \left[ r_P(t_1; t_{i0}) \right] = \equiv \left[ r_{SU}^i(t_1; t_{i0}) \right]^T G \left[ r_{SU}^i(t_1; t_{i0}) \right] = \equiv \left[ v_{SU}^\prime O_{SU}^i \mu_i(t - t_{0}) \right]^T G \left[ v_{SU}^\prime O_{SU}^i \mu_i(t - t_{0}) \right] = \equiv \left[ v_{SU}^\prime O_{SU}^i (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right]^T G \left[ v_{SU}^\prime O_{SU}^i (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right] = \equiv \left[ \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right]^T G \left[ \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right] = \equiv \left[ \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right]^T G \left[ \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right] = \equiv \left[ \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right]^T G \left[ \lambda_i^j (v_{SU}^\prime + v_{j}^\prime) \mu_j (t - t_{0}) \right].
\]

These identities imply:

\[
\left[ \mu_j v_{SU}^\prime \right]^{-1} = \equiv 1.
\]

This proves the first equality in (12). Since the point \( P_{SU} \) is arbitrary, then it can represent the light signal so that then \( v_{SU}^\prime = c_{ij}^\prime \), \( \lambda = i, j \), which implies the second equation in (12) in view of the preceding result. For the same reason, the third and the fourth equation in (12) hold as soon as \( v_{P}^\prime \in R^+ \) and \( v_{j}^\prime \in R^+ \). The equations (18) and (19) yield:

\[
\lambda_i^j \equiv \left[ 1 + \frac{v_{j}^\prime}{v_{SU}^\prime} \right]^{-1}.
\]

This proves the equation (9). The equation (10) is proved in the same manner. The equations (7) through (10), real values of the scaling coefficients, and the definition of \( v_{j}^\prime \) imply the inequalities in (11). The first equations in (7) and (8) together with (12) result, respectively, in the second equations in (7) and (8). The equations (7) through (10) transform (2) through (5) into (13) through (16).

**Sufficiency.** Let the spatial reference point \( P_{SU} \) be freely chosen and fixed. Let it move with a known constant speed \( v_{SU}^\prime \in R^+ \). Let the arbitrary point \( P \) be \( P_{SU} \),
\[ v^j_i = v^j_{SU}. \] Let time scaling coefficients \( \mu_k \in R^+ \), \( k = 1, 2, \ldots, s \), be defined by (1). Let (7) – (12) and all the conditions of the theorem statement hold. We start with (1), (7) and (12):
\[
t^i - t^0 = \mu_i (t - t^0) j^i \langle j^i \rangle^{-1} =
\[
= \mu_i (t - t^0) j^i \left( 1 + \frac{v^j_i v^j_{SU}}{q^j \omega^j} \right).
\]
or, by using (1) for \( k = j \), and \( r_p(t_i; t_j) =
\[
= v^j_i j^i \left( 1 + \frac{v^j_i v^j_{SU}}{q^j \omega^j} \right)(t_j - t^0) =
\[
= \alpha^j \left( t_j - t^0 \right) + \frac{v^j_i}{q^j \omega^j} r_p(t_j; t^0).
\]
This proves the validity of (2). The equation (3) is proved analogously by starting with \( t_j - t^0 \). We transform \( r_p(t_i; t^0) = v^0_p j^i \langle j^i \rangle^{-1} (t_i - t^0) = v^0_p \langle j^i \rangle^{-1} (t_i - t^0) \) by using \( r_p(t_i; t^0) = v^0_p \langle j^i \rangle^{-1} (t_i - t^0) = v^0_p (t_i - t^0) \). (1), (9) and (12):
\[
r_p(t_i; t^0) = v^0_p (t_i - t^0) \mu_i (t_i - t^0) j^i \langle j^i \rangle^{-1} (t_i - t^0) =
\[
= v^0_p \lambda^i \left( 1 + \frac{v^j_i}{v^0_p} \right) j^i \langle j^i \rangle^{-1} (t_i - t^0) =
\[
= v^0_p \lambda^i \left( 1 + \frac{v^j_i}{v^0_p} \right) \mu_i (t_i - t^0) j^i \langle j^i \rangle^{-1} (t_i - t^0) =
\[
= v^0_p \lambda^i \left( 1 + \frac{v^j_i}{v^0_p} \right) \mu_i (t_i - t^0) j^i \langle j^i \rangle^{-1} (t_i - t^0) =
\[
= v^0_p \lambda^i \left( 1 + \frac{v^j_i}{v^0_p} \right) \mu_i (t_i - t^0) j^i \langle j^i \rangle^{-1} (t_i - t^0) =
\[
= v^0_p \lambda^i \left( 1 + \frac{v^j_i}{v^0_p} \right) (t_i - t^0) =
\]
This proves (4). The equation (5) is analogously proved by starting with
\[
r_p(t_i; t^0) = v^0_p j^i \langle j^i \rangle^{-1} (t_i - t^0) = v^0_p (t_i - t^0) =
\]
Let us now transform the left-hand side of (6) as follows by using
\[
r_p(t_i; t^0) = v^0_p (t_i - t^0) = v^0_p (t_i - t^0) =
\]
...
This proves (6) and completes the proof of sufficiency.

**Compatibility.** We verify compatibility of the transformations (13) through (16) as follows. We allow arbitrary speed \( v_j^i \in \mathbb{R}^+ \) of \( P \). We replace \( (t_j - t_0) \) from (14) into (13):

\[
t_j - t_0 = \left[ \frac{\mu_j}{\mu_i} \left( t_j - t_0 \right) - \frac{v_j^i}{q^i_0} r_P(t_j; t_0) \right] \left[ 1 - \frac{v_j^i v_SU}{q^i_0} \right] + \frac{v_j^i}{q^i_0} r_P(t_j; t_0)
\]

\[
\cdot \left( 1 + \frac{v_j^i v_SU}{q^i_0} \right)^{-1}
\]

and we use \( r_P(t_j; t_0) = v_P^j(t_j - t_0) \) for \( (.) = i, j, \) and (1),

\[
t_j - t_0 = \left[ \frac{\mu_j}{\mu_i} \left( t_j - t_0 \right) - \frac{v_j^i}{q^i_0} \left( t_j - t_0 \right) \right] \left[ 1 - \frac{v_j^i v_SU}{q^i_0} \right] + \frac{v_j^i}{q^i_0} \left( t_j - t_0 \right)
\]

\[
\cdot \left( 1 + \frac{v_j^i v_SU}{q^i_0} \right)^{-1}
\]

For these identities to hold it is necessary and sufficient that \( v_P^j = v_SU^j \). Hence, (13) and (14) are only partially compatible. In order to verify compatibility of the spatial co-ordinate transformations we replace at first \( r_P(t_j; t_0) \) by the right-hand side of (16) into (15), we use \( r_P(t_j; t_0) = v_P^j(t_j - t_0) \) and afterwards we apply (1) and (12):

\[
r_P(t_j; t_0) = \frac{r_P(t_j; t_0) - v_P^j(t_j - t_0) u}{1 - \frac{v_P^j}{O_j^{SU}}} + \frac{v_P^j(t_j - t_0) u}{v_P^{SU}}
\]

\[
r_P(t_j; t_0) = \frac{1 - \frac{v_j^i}{O_j^{SU}}}{1 + \frac{v_j^i}{O_j^{SU}}}
\]

For these identities to hold it is necessary and sufficient that \( v_P^j = v_SU^j \). Hence, (15) and (16) are only partially compatible. Altogether, the transformations (13) through (16) are partially pairwise compatible. We will test their complete entire compatibility as follows. We replace, respectively, \( (t_j - t_0) \) and \( r_P(t_j; t_0) \) from (13) and (16) into (15):

\[
r_P(t_j; t_0) = \left[ 1 + \frac{v_j^i}{O_j^{SU}} \right]^{-1}
\]

\[
\cdot \left[ \frac{\mu_j}{\mu_i} \left( t_j - t_0 \right) - \frac{v_j^i}{q^i_0} r_P(t_j; t_0) \right] \left[ 1 - \frac{v_j^i v_SU}{q^i_0} \right] + \frac{v_j^i}{q^i_0} \left( t_j - t_0 \right)
\]

\[
\cdot \left( 1 + \frac{v_j^i v_SU}{q^i_0} \right)^{-1}
\]

\[
\cdot \left[ 1 - \frac{v_j^i v_SU}{q^i_0} \right] + \frac{v_j^i}{q^i_0} \left( t_j - t_0 \right)
\]

\[
\cdot \left( 1 + \frac{v_j^i v_SU}{q^i_0} \right)^{-1}
\]

\[
\cdot \left[ 1 + \frac{v_j^i}{O_j^{SU}} \right]^{-1}
\]

\[
\cdot \left[ \frac{\mu_j}{\mu_i} \left( t_j - t_0 \right) - \frac{v_j^i}{q^i_0} r_P(t_j; t_0) \right] \left[ 1 - \frac{v_j^i v_SU}{q^i_0} \right] + \frac{v_j^i}{q^i_0} \left( t_j - t_0 \right)
\]

\[
\cdot \left( 1 + \frac{v_j^i v_SU}{q^i_0} \right)^{-1}
\]

\[
\cdot \left[ 1 - \frac{v_j^i v_SU}{q^i_0} \right] + \frac{v_j^i}{q^i_0} \left( t_j - t_0 \right)
\]

\[
\cdot \left( 1 + \frac{v_j^i v_SU}{q^i_0} \right)^{-1}
\]
The equations (13) through (16) are beyond the Lorentz - Einstein relativity theory. The former are rational functions of \( v^i_{ji} / q^i \), \( v^j_{ji} / q^j \), \( v^i_{ji} / v^O_P \), or \( v^j_{ji} / v^O_P \), while the latter are not. The former cannot contain square roots or squared speeds, while the latter do both. The former contain the relative values of all speeds, but not the latter. The former are consistent, but the latter are inconsistent. The former are partially pairwise and entirely compatible, while the latter are completely entirely compatible. The former do not contain the light speed in general, while the latter do necessarily. Therefore, the former show that the light speed is not a necessary generic speed, while the latter demand that exclusively.

The preceding theorem results from the characterisation of \textit{time} based on the physical reality and our experience (Axiom 1). The \textit{time} independence is expressed by different scaling factors in equations (2) through (5), and by \( \Lambda \neq B \) in G, (6). The obtained scaling coefficients, (7) through (10), differ essentially from those by Lorentz. The above result is general.

The special case will be examined in the sequel.

The preceding theorem takes the following form in the case the reference frames \( R_i^n \) and \( R_j^n \) move in parallel with the same velocity: \( v^i_{ji} = v^j_{ji} = 0 \). This is important for dynamical systems (S1), (S2).

\begin{equation}
\alpha_j^i = \frac{\mu_j}{\mu_i} = \frac{v^j_{SU}}{v^i_{SU}} = \frac{c_j^i}{c_i^j}, \quad \alpha_j^i = \frac{\mu_j}{\mu_i} = \frac{v^j_{SU}}{v^i_{SU}} = \frac{c_j^i}{c_i^j}, \quad \lambda_j^i = \lambda^i_j = 1, \quad i, j = 1, 2, \ldots, n.
\end{equation}

The coordinate transformations (2) through (5) reduce to
\begin{equation}
t_i - t_{i0} = \frac{\mu_j}{\mu_i} \left( t_j - t_{j0} \right), \quad r_p(t_i; t_{i0}) = r_p(t_j; t_{j0})
\end{equation}

The (numerical) value of the light speed need not be the same with respect to \( I_i = T_i \times R_i^n \) and \( I_j = T_j \times R_j^n \) as soon as the \textit{time} scales of \( T_i \) and \( T_j \) are different:
\( \mu_i \neq \mu_j \).

The space scaling factors \( \lambda_j^i = \lambda^i_j = 1 \) because \( R_i^n \) and \( R_j^n \) move with the same speed.

The preceding corollary is significant for dynamical systems (S1), (S2) with multiple \textit{time} scales. It links the relativity theory with the theory of these systems. It establishes the conditions under which the dynamical system (S1), (S2) can have different \textit{time} scales.

\subsection*{7.3 Problem solution for the special case}

In the case the relative values \( c_i^i \) and \( c_j^i \) of the light speed are mutually equal, \( c_i^i = c_j^i \), then they are denoted by \( c^i \) or by \( c^i, c_i^i = c_j^i = c^i = c^j \). This designates the same relative value of the light speed with respect to \( I_i \) and \( I_j \). Then, \( v_{ji} \) denotes that \( v^i_{ji} \) and \( v^j_{ji} \) are mutually equal:
\begin{equation}
v^i_{ji} = v^j_{ji} = v_{ji} = -v_{ij} = -v^i_{ij} = -v^j_{ij}.
\end{equation}

Analogously,
\( v_{SU}^i = v_{SU}^j \Rightarrow v_{SU}^i = v_{SU}^j = v_{SU}^{ij} \)

\( q^i \omega^i = q^j \omega^j \Rightarrow q^i \omega^i = q^j \omega^j = (q^i)^j = (q^j)^i \).

**Theorem 2.** Let the time scaling coefficients \( \mu_i \in R^+ \), \( i = 1, 2, \ldots, s \), be defined by (1). Let the spatial reference point \( P \) be arbitrarily chosen and then fixed moving with a known constant velocity \( \mathbf{v}_{SU}^i = \mathbf{v}_{SU}^j \mathbf{u} \), \( \mathbf{v}_{SU}^i \in R^+ \). If the speed of the arbitrary point \( P \) is arbitrary, then, in order for the scaling coefficients \( \alpha_i^j \in R^+ \), \( \alpha_i^j \in R^+ \), \( \alpha_i^{ij} = \alpha_i^j = \alpha_{ij} = \alpha_{ji} \), \( \lambda_i^j \in R^+ \) and \( \lambda_i^{ij} = \lambda_{ij} = \lambda_{ji} \), determined for the speed \( \mathbf{v}_{SU}^i \) of \( P \), to be positive real numbers and to obey (2) through (5), and for (1) through (5) to imply (6) it is necessary (but not sufficient) that the following equations hold for any choice of the time scaling coefficient \( \mu_i \in R^+ \):

\[
O^{ij}_{SI} = \sqrt{q^i \omega^i} \omega^j, \quad i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
v_{ji}^{ij} = \frac{v_{ji}^i}{v_{ji}^j} = \frac{c_i}{c_j}, \quad i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
\alpha_i^j = \frac{1}{1 - \left( \frac{v_{ji}^i}{q^i \omega^i} \right)^2} = \frac{1}{1 - \left( \frac{v_{ji}^j}{q^j \omega^j} \right)^2} = \alpha_i^j,
\]

\[
\lambda_i^j = \frac{1}{1 - \left( \frac{v_{ji}^i}{v_{ji}^j} \right)^2} = \frac{1}{1 - \left( \frac{v_{ji}^j}{v_{ji}^i} \right)^2} = \lambda_i^j,
\]

\[
u_{ij}^i \mu_j = \mu_i \frac{v_{ij}^i}{v_{ij}^j} \frac{v_{ij}^j}{v_{ij}^i} = \mu_i \left( \frac{v_{ij}^i}{v_{ij}^j} \right)^2,
\]

\[
i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
\lambda_i^j = \frac{v_{ji}^i}{v_{ji}^j} = \frac{c_i}{c_j}, \quad i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
\frac{v_{ji}^i}{v_{ji}^j} < v_{ij}^i < v_{ij}^j \quad \text{for (}i,j), \quad i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
u_{ij}^i = \frac{v_{ji}^i}{v_{ji}^j} = v_{ji}^i. \quad \text{for (}i,j), \quad i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

The equations (2) through (5) become the equations (27) through (30),

\[
t_j - t_{j0} = \sqrt{\frac{1 - \left( \frac{v_{ji}^i}{q^i \omega^i} \right)^2}{1 - \left( \frac{v_{ji}^j}{q^j \omega^j} \right)^2}}, \quad i,j \in \{-1, 1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
r_p(t_j; t_{j0}) = r_p(t_j; t_{j0}) + v_{ji}^i(t_j - t_{j0})/u,
\]

\[
r_p(t_j; t_{j0}) = r_p(t_j(t_{j0}) - v_{ji}^j(t_j - t_{j0})/u,
\]

\[
i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j.
\]

If additionally \( A = B \) in \( G \), (6), then for the coefficients \( \alpha_i^j \in R^+ \), \( \alpha_i^{ij} = \alpha_i^j = \alpha_{ij} = \alpha_{ji} \), \( \lambda_i^j \in R^+ \) and \( \lambda_i^{ij} = \lambda_{ij} = \lambda_{ji} \), determined for the speed \( v_{SU}^i \) of \( P \), to be positive real numbers and to obey (2) through (5), and for (1) through (5) to imply (6) it is necessary and sufficient that the equations (20) – (26) and the following equations hold for any choice of the time scaling coefficient \( \mu_i \in R^+ \):

\[
v_{ji}^i = v_{ji}^j = v_{ji}^i. \quad \text{for (}i,j), \quad i,j \in \{-1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
v_{ji}^i = v_{ji}^j = v_{ji}^i = \sqrt{q^i \omega^i} = \sqrt{q^j \omega^j} = \sqrt{(q^i)^j} = (q^i)^j.
\]

\[
i,j \in \{-1, 1, 2, \ldots, s\}, \quad i \leq j,
\]

\[
c_i = c_j = c_i \cdot i,j \in \{-1, 1, 2, \ldots, s\}, \quad i \leq j.
\]
The transformations (27) through (30) are partially both
pairwise and entirely compatible.

**Proof.** The proof of this theorem is very long. It is
therefore omitted due to the space limitation. Its
methodology is the same as that of the proof of
Theorem 2 in [21]. It is worked out in [25].

The equations (22), (23) for the scaling factors
contain consistently the relative values of all the speeds,
therefore omitted due to the space limitation. Its
methodology is the same as that of the proof of
Theorem 2 in [21]. It is worked out in [25].

Notice that for $A \neq B$ in G, (6), the above theorem
provides only the necessary, but not the sufficient,
conditions.

**Note 1.** If $A = B$ in G, (6), then the equations (27)
and (28) can be set into the following forms:

\[

t_j + \frac{v_{ji}}{v_{ji}^R} r_p(t_j; t_{j0}) = \frac{t_i - t_{i0}}{1 - \left(\frac{v_{ji}}{v_{ji}^R}\right)^2},
\]

\[i, j \in \{1, 2, \ldots, s\}, i \leq j,
\]

\[t_i - t_{i0} + \frac{v_{ji}}{v_{ji}^R} r_p(t_i; t_{i0}) = \frac{t_j - t_{j0}}{1 - \left(\frac{v_{ji}}{v_{ji}^R}\right)^2},
\]

\[i, j \in \{1, 2, \ldots, s\}, i \leq j.
\]

(27b)

(28b)

7.4 Problem solutions for the singular cases

In the singular cases the Constraint 1 through the
Constraint 6 are valid. They give the following form to
the Theorem 2:

**Corollary 2.** Let the Constraint 1 through the
Constraint 3, and the Constraint 6 be valid. Let the time
scaling coefficients $\mu_i \in R$, $i = 1, 2, \ldots, s$, be defined by (1). Let the spatial reference point $R_{SU}$ be arbitrarily
chosen and then fixed moving with a known constant
velocity $v_{SU}^R \in R^+$, $v_{SU}^R \in R^+$. In order for the
coefficients $\alpha_j^i \in R$, $\alpha_j^i \in R$, $\alpha_j^i = \alpha_j^i = \alpha_j^i = \alpha$, $\lambda_j^i \in R^+$ and $\lambda_j^i \in R^+$,
$\lambda_j^i = \lambda_j^i = \lambda$, determined for the speed $v_{SU}^R$ of $R_{SU}$, to be positive real numbers and to obey (2)
through (5), and for (1) through (5) to imply (6) it is both
necessary and sufficient that the equations (20)
through (30) hold for any choice of the time scaling
coefficient $\mu_i \in R$, for $v_{SU} = v_{SU} = v_{SU} = c$ and for
$v_{j} = v$. The equations (2) through (5) become the Non-
Lorentz transformations (NL1) through (NL4):

\[\begin{align*}
(t_j - t_{j0}) + \frac{v}{v_{SU}} r_p(t_j; t_{j0}) &= \frac{t_i - t_{i0}}{1 - \left(\frac{v}{v_{SU}}\right)^2}, & (NL1) \\
(t_i - t_{i0}) - \frac{v}{v_{SU}} r_p(t_i; t_{i0}) &= \frac{t_j - t_{j0}}{1 - \left(\frac{v}{v_{SU}}\right)^2}, & (NL2) \\
r_p(t_i; t_{i0}) = r_p(t_j; t_{j0}) + v(t_j - t_{j0}) &= \frac{1}{1 - \left(\frac{v}{v_{SU}}\right)^2}, & (NL3) \\
r_p(t_j; t_{j0}) = r_p(t_i; t_{i0}) - v(t_i - t_{i0}) &= \frac{1}{1 - \left(\frac{v}{v_{SU}}\right)^2}. & (NL4)
\end{align*}\]

The transformations (NL1) through (NL4) are partially
both pairwise and entirely compatible.

The Non-Lorentz transformations (NL1) through
(NL4) incorporate the Lorentz transformations as a
singular case. The former reduce to the latter when the
spatial reference point moves with the light speed $c$, i.e.
for $v_{SU} = c$. The following Corollary shows this
explicitly:

**Corollary 3.** Let Constraint 1 through Constraint 6 be
valid. Let the time scaling coefficients $\mu_i \in R$, $i = 1, 2, 
\ldots, s$, be defined by (1). In order for the coefficients
$\alpha_j^i \in R^+$, $\alpha_j^i \in R^+$, $\alpha_j^i = \alpha_j^i = \alpha_j^i = \alpha$, $\lambda_j^i \in R^+$ and
$\lambda_j^i \in R^+$, $\lambda_j^i = \lambda_j^i = \lambda_j^i = \lambda$, determined for the light
speed $c_{ij}$ of $P_{SU}$, to be positive real numbers and to obey
(2) through (5), and for (1) through (5) to imply
(6) it is both necessary and sufficient that the equations
(20) through (30) hold for any choice of the time scaling
coefficient $\mu_i \in R^+$, for $v_{SU} = v_{SU} = v_{SU} = c$ and for
$v_{j} = -v$.

The equations (2) through (5) become the Lorentz transformations (L1) through (L4):
Let us repeat once more, a change of the speed of the clock hands means nothing else than the corresponding change of the time unit. Once we determine the scaling coefficients among the time scales corresponding to different speeds of the clock hands, then we can use the equation (1) to verify that all the clocks show always the same moment, provided they are identical, operate identically and perfectly, accurately and precisely, in the same conditions, their hands were set at the same initial position and they started to operate at the same instant. The book [25] exposes more details and proofs in this regard.

Comment 1. It is not time that causes a variation of the kinetic energy of a moving body in general, i.e. of the mass of the moving body in the Lorentzian inertial frames, when the body speed varies. It is a variation of the energy exchange between the body and its environment, which causes a variation of the kinetic energy of a moving body in general, hence, which causes the variation of the mass of the body in the Lorentzian inertial frames. For the energy - mass explanation see the papers by Marmet [35] - [40]. The book [25] presents the general kinematic results in this connection.

8. VELOCITY TRANSFORMATIONS

8.1. General case

Theorem 3. Let the time scaling coefficients \( \mu_i \in \mathbb{R}^+ \), \( i = 1, 2, \ldots, s \), be defined by (1). Let the scaling coefficients \( \mu_i, \alpha^i_j \in \mathbb{R}^+, \alpha^i_j \neq \alpha^i_j, \lambda^i_j \in \mathbb{R}^+, \lambda^i_j \neq \lambda^i_j \), be positive real numbers and obey (2) through (5), and let (1) through (5) imply (6). Then, a constant non-zero velocity \( \mathbf{v}^i_{P_1} \) of the arbitrary point P with respect to the origin \( O_i \) of \( R^i_{O_1} \) and relative to \( t_i \) and the velocity \( \mathbf{v}^i_{P_1} \) of the same point P with respect to the origin \( O_j \) of \( R^j_{O_1} \) and relative to \( t_j \) are interrelated as follows:

\[
\mathbf{v}^j_{P} = \frac{1}{\mu_j} \left[ \frac{\mathbf{v}^j_{SU} + \mathbf{v}^j_{O_1}}{1 + \frac{\mathbf{v}^j_{SU}}{q_j \omega_j}} \right] = \frac{1}{\mu_j} \frac{\mathbf{v}^j_{SU}}{1 - \frac{\mathbf{v}^j_{SU}}{q_j \omega_j}} \mathbf{v}^j_{P} + \mathbf{v}^j_{ji},
\]

\[
\mathbf{v}^i_{P_1} = \frac{1}{\mu_i} \left[ \frac{\mathbf{v}^i_{SU} + \mathbf{v}^i_{O_1}}{1 + \frac{\mathbf{v}^i_{SU}}{q_i \omega_i}} \right] = \frac{1}{\mu_i} \frac{\mathbf{v}^i_{SU}}{1 - \frac{\mathbf{v}^i_{SU}}{q_i \omega_i}} \mathbf{v}^i_{P_1} - \mathbf{v}^i_{ji}.
\]

In order for them to be compatible it is necessary and sufficient that

\[
\mathbf{v}^i_{SU} \cdot \mathbf{v}^j_{O_1} = \mathbf{v}^j_{SU} \cdot \mathbf{v}^i_{P_1} = \mathbf{v}^j_{ji}.
\]

The transformations are partially compatible.
Proof. Let all the conditions be valid. Hence, the Theorem 1 holds. The velocity is defined as:

\[ v_{P}^{O_{j}}(t_{ij}) = \frac{d\tau_{ij}(t_{ij}, t_{j0})}{d\tau_{ij}} = v_{j}^{O_{j}}. \] (33)

By applying (13) and (15) to the right hand side of the preceding equation and by using (1),
\[ dr_{P}(t_{ij}; t_{j0})/dt_{j} = v_{P}^{O_{j}}, \quad v_{ji}^{O_{j}} = v_{ji}^{O_{i}} \] and (12) we find:

\[ v_{P}^{O_{j}} = \frac{d}{dt_{j}} \left[ \frac{\mu_{i} (t_{j} - t_{j0}) v_{ji}^{O_{j}} + v_{ji}^{O_{j} v_{ji}^{O_{j}}}}{\mu_{j} 1 + \frac{v_{ji}^{O_{j}} v_{ji}^{O_{j}}}{\sqrt{1 - v_{ji}^{O_{j} v_{ji}^{O_{j}}}}} \right] = 0 \]

Then
\[ \frac{v_{ij}^{O_{j}} + v_{ji}^{O_{j}} u}{1 + \frac{v_{ji}^{O_{j}}}{\sqrt{1 - v_{ji}^{O_{j} v_{ji}^{O_{j}}}}} = \frac{\mu_{j} v_{ji}^{O_{j}}}{\mu_{i} 1 + \frac{v_{ji}^{O_{j}} v_{ji}^{O_{j}}}{\sqrt{1 - v_{ji}^{O_{j} v_{ji}^{O_{j}}}}} \right] = 0 \]

This proves (31). The equation (32) is analogously proved by using (1), (12), (14), (16), (33),
\[ dr_{P}(t_{ij}; t_{j0})/dt_{i} = v_{j}^{O_{j}} \] and \( v_{ji}^{O_{j}} u = v_{ji}^{O_{j}} \). It is easy to check that for (31) and (32) to be compatible it is necessary and sufficient that \( v_{P}^{O_{j}} = v_{ji}^{O_{j}} \) for \( v_{ji}^{O_{j}} = v_{ji}^{O_{j}} \). Q. E. D.

The equations (31) and (32) confirm the equations (12).

The velocity transformation equations (31) and (32) are beyond the Lorentz – Poincaré - Einstein relativity theory. The speed value ratios in the denominators of the right hand side quotients in (31) and (32) do not contain the light speed value, which appears in the ratio \( c_{v_{ji}^{O_{j}}} / c \) in the velocity transformations resulting from the Lorentz transformations, [30] - [32]. If we accept \( v_{ji}^{O_{j}} = c_{ji}^{O_{j}} \), then the formulas (31) and (32) contain the light speed values \( c_{ji}^{O_{j}} \) and \( c_{ji}^{O_{j}} \) relative to \( I_{j} \) and \( I_{j} \), respectively. However, the light speed value is independent of time scale and of time unit in the Lorentz – Poincaré - Einstein relativity theory [3] – [7], [30] - [33], [50].

Note 2. The equations (31) and (32) show that
- the light speed is not invariant in general,
- the light speed value does not restrict the values of other speeds,
- the light speed obeys the general rules,
- the light speed is not an exceptional speed from the kinetic point of view.

8.2. Special case

Theorem 4. Let the time scaling coefficients \( \mu_{i} \in R^{i}, i = 1, 2, \ldots, s \), be defined by (1). Let \( \Lambda = B \) in G. Let the scaling coefficients \( \alpha_{i}^{j} \in R^{i}, \alpha_{j}^{i} \in R^{i}, \alpha_{i}^{j} = \alpha_{j}^{i} = \alpha_{ij} = \alpha_{ji}, \lambda_{ij}^{j} \in R^{j} \) and \( \lambda_{ji}^{j} \in R^{i} \). \( \lambda_{ij}^{j} = \lambda_{ji}^{j} \), be positive real numbers and obey (2) through (5), and let (1) through (5) imply (6). Then, a constant non-zero velocity \( v_{P}^{O_{j}} \) of the arbitrary point P with respect to the origin \( O_{j} \) of \( R_{j}^{n} \) and relative to \( t_{i} \), the velocity \( v_{P}^{O_{j}} \) of the same point P with respect to the origin \( O_{j} \) of \( R_{j}^{n} \) and relative to \( t_{i} \) are interrelated as follows:

\[ v_{P}^{O_{j}} = \frac{v_{ji}^{O_{j}} + v_{ji}^{O_{j}} u}{1 + \frac{v_{ji}^{O_{j}}}{\sqrt{1 - v_{ji}^{O_{j} v_{ji}^{O_{j}}}}} = \frac{\mu_{j} v_{ji}^{O_{j}}}{\mu_{i} 1 + \frac{v_{ji}^{O_{j}} v_{ji}^{O_{j}}}{\sqrt{1 - v_{ji}^{O_{j} v_{ji}^{O_{j}}}}} \right] = 0 \]

Proof. Let all the conditions hold. Hence, the Theorem 2 and Note 1 are valid. The equations (27b), (28b), (29), (30) and (33) furnish (34). Q. E. D.

Note 3. When we accept \( v_{P}^{(1)} = c_{P}^{(1)} \) in (34) then it results that the light speed is not invariant, but it obeys the general rule as all other speeds.

Theorem 5. Let the time scaling coefficients \( \mu_{i} \in R^{i}, i = 1, 2, \ldots, s \), be defined by (1). Let \( \Lambda = B \) in G. Let the scaling coefficients \( \alpha_{i}^{j} \in R^{i}, \alpha_{j}^{i} \in R^{i}, \alpha_{i}^{j} = \alpha_{j}^{i} = \alpha_{ij} = \alpha_{ji}, \lambda_{ij}^{j} \in R^{j} \) and \( \lambda_{ji}^{j} \in R^{i} \). \( \lambda_{ij}^{j} = \lambda_{ji}^{j} \), be positive real numbers and obey (7) through (10), and let (1) through (5) imply (6). Then,
a) The spatial reference speed \( v_{SU}^{(i)} \) is invariant relative to the integral spaces \( I_i \) and \( I_j \), \( v_{SU}^{(i)} = v_{SU}^{(j)} = v_{ij}^{SU} \).

b) For the light speed \( c_{ij}^{(i)} \) to be invariant relative to the integral spaces \( I_i \) and \( I_j \), \( c_{ij}^{(i)} = c_{ij}^{(j)} = c_{ij}^{SU} \), it is necessary and sufficient to be the spatial reference speed \( v_{SU}^{(i)} \), \( c_{ij}^{(i)} = v_{SU}^{(i)} = v_{ij}^{SU} \).

**Proof.** Let all the conditions be satisfied.

a) Let the arbitrary point \( P \) move with the spatial reference velocity, \( v_{P}^{(i)} = v_{SU}^{(i)} \). Then the equations (34) become the following:

\[
v_{P}^{O(\mu)} = v_{SU}^{i} = \frac{v_{SU}^{j} + \frac{v_{ij}^{SU}}{1 + \frac{v_{ij}^{SU}}{v_{ij}^{SU}}}}{1 + \frac{v_{ij}^{SU}}{v_{ij}^{SU}}},
\]

\[
v_{P}^{O(j)} = v_{SU}^{j} = v_{SU}^{i},
\]

Hence, \( v_{SU}^{(i)} = v_{SU}^{(j)} = v_{ij}^{SU} \).

b) For the light speed to be invariant in view of (34) it is necessary and sufficient that

\[
v_{P}^{O(\mu)} = c_{ij}^{(i)} = c_{ij}^{(j)} - c_{ij}^{SU}, \quad \text{and}
\]

\[
v_{P}^{O(j)} = c_{ij}^{(j)} = c_{ij}^{SU},
\]

For these equations to hold it is necessary and sufficient that \( c_{ij}^{(i)} = v_{SU}^{(i)} = v_{ij}^{SU} = c_{ij}^{SU} = c_{ij}^{(j)} \).

Q. E. D.

8.3. **Singular case**

**Theorem 6.** Let the time scaling coefficients \( \mu_i \in R^+, i = 1, 2, \ldots, s \), be defined by (1). Let \( A = B \) in G. Let \( v_{ij}^{(i)} = v \) and \( v_{SU}^{(i)} = v_{SU} \). Let the scaling coefficients \( \mu_i, \alpha_{ij}^{(i)} \in R^+, \alpha_{ij}^{(j)} \in R^+, \alpha_{ij}^{(i)} = \alpha_{ij}^{(j)} = \alpha, \lambda_{ij}^{(i)} \in R^+ \) and \( \lambda_{ij}^{(j)} \in R^+, \lambda_{ij}^{(j)} = \lambda \), be positive real numbers and obey (2) through (5), and let (1) through (5) imply (6). Then, a constant non-zero speed \( v_{P}^{O(\mu)} \) of the arbitrary point \( P \) with respect to the origin \( O \), of \( R^\mu \) and relative to \( t_i \) and the speed \( v_{P}^{O(\mu)} \) of the same point \( P \) with respect to the origin \( O_j \), of \( R_j^\mu \) and relative to \( t_i \), are interrelated as follows:

\[
v_{P}^{O(\mu)} = \frac{v_{P}^{O(\mu)} + v}{1 + \frac{v_{ij}^{SU}}{v_{ij}^{SU}}},
\]

\[
v_{P}^{O(j)} = \frac{v_{P}^{O(j)} - v}{1 - \frac{v_{ij}^{SU}}{v_{ij}^{SU}}},
\]

**Proof.** Let all the conditions be satisfied. Then the Corollary 2 holds. We replace the equations (27b), (28b), (29), (30) by the equations (NL1) through (NL4) in the proof of the Theorem 5, which then becomes the proof of the Theorem 6, Q. E. D.

**Theorem 7.** Let the time scaling coefficients \( \mu_i \in R^+, i = 1, 2, \ldots, s \), be defined by (1). Let \( A = B \) in G. Let the scaling coefficients \( \mu_i, \alpha_{ij}^{(i)} \in R^+, \alpha_{ij}^{(j)} \in R^+, \alpha_{ij}^{(i)} = \alpha_{ij}^{(j)} = \alpha, \lambda_{ij}^{(i)} \in R^+ \) and \( \lambda_{ij}^{(j)} \in R^+, \lambda_{ij}^{(j)} = \lambda \), be positive real numbers and obey (2) through (5), and let (1) through (5) imply (6). Let \( v_{ji}^{(i)} = v_{ji}^{(j)} \). Then,

c) The spatial reference speed \( v_{SU}^{(i)} \) is invariant relative to all integral spaces, \( v_{SU}^{(i)} = v_{SU} \).

d) For the light speed \( c_{ij}^{(i)} \) to be invariant relative to all integral spaces, \( c_{ij}^{(i)} = c \), it is necessary and sufficient to be the spatial reference speed \( v_{SU}^{(i)} \), \( c_{ij}^{(i)} = v_{SU}^{(i)} = v_{SU} \).

**Proof.** Under the theorem statement, the Corollary 2 is valid. When we set \( v_{ji}^{(i)} = v_{ji}^{(j)} \) in the proof of the Theorem 5 then it becomes the proof of the Theorem 7 since the scaling coefficients and are the same for all integral spaces mutually related by the transformations (27) through (30), which reduce in this case to (NL1) through (NL4), and for the light speed of the reference point they reduce to (L1) through (L4), Q. E. D.

**Corollary 4.** a) For every speed there are co-ordinate and velocity transformations under which the velocity is invariant.

b) For the light speed such transformations are the Lorentz transformations (L1) through (L4), and from them resulting Einstein's law of the composition of velocities.
Note 4. The light speed is not an exceptional speed. It obeys the general rules.

9. CONCLUSION

Our experience with, our understanding of, and our knowledge about the physical reality led to the axiomatic characterisation of the properties of time (Axiom 1). It is in Newton's sense. It expresses that time is a unique and a basic physical variable. There are not two or more different times. There are many different time scales and time units. Newton himself explained this in [48]. The numerical value of the speed of time value evolution (of the time speed) equals one (with respect to all time axes). It is invariant relative to a choice of a time axis [25]. It is independent of everybody and everything, hence, of the space. Time is the unique physical variable with such property of its speed [25].

The numerical value of the speed of light is not invariant with respect to a choice of a time axis.

The features of time enabled us to avoid all the constraints a priori accepted in the Einsteinian relativity theory after Lorentz and Einstein, and to start establishing fundamentals of a new relativity theory.

The co-ordinate transformations are compatible if, and only if the application of the inverse transformation to the transformation itself results in an identity. If, and only if this holds for both pairs, the pair of the temporal co-ordinate transformations and the pair of the spatial co-ordinate transformations, then the transformations are pairwise compatible. If, and only if this is true for the temporal and spatial transformations altogether, then they are entirely compatible. If, and only if this compatibility of the transformations is valid only for the light speed of an arbitrary point, or under restriction on the generic speeds, then the compatibility of the transformations is partial (restrictive) compatibility. Their compatibility is complete if, and only if it holds for arbitrary value of the speed of an arbitrary point, and, if there is not any restriction on the generic speeds.

Uniformity of the transformations means that the temporal co-ordinate transformations are independent of a choice of an arbitrary point (of its position and of its speed), i.e., that they hold uniformly over the space. Uniform co-ordinate transformations were established in [20] through [23], [25]. Otherwise, the transformations are non-uniform, which holds for the transformations (13) through (16) established herein.

The time scaling factors determined by (7) through (10), or by (22), (23), and the related new co-ordinate transformations (13) through (16), respectively, (27 through (30), do not contain the speed of light in general.

It is shown that starting with the Axiom 1 and by accepting Constraint 1 through Constraint 6, we get the Lorentz co-ordinate transformations. This clarifies that they may not, and cannot be used to claim either time dependence on the space or a variation of the speed of time value evolution. Such claims are wrong. Time does not force any clock to work. A variation of the speed of clock hands means only the corresponding variation of the time units, and vice versa. It does not mean either a variation of the speed of time value evolution or the existence of different times for clocks moving with different speeds. Energy, not time, forces the clock to operate.

The velocity transformations (31), (32) in the general case, (34) in the special case, (35) in the singular case, are essentially different from Einstein's velocity transformations known as Einstein's law of the velocity compositions.

The Theorems 5 and 7 discover that the light speed is not an exceptional speed. It is not invariant and it does not restrict other speeds, in general. The Theorem 7 shows also that for every speed we can construct co-ordinate transformations, and from them resulting velocity transformations, so that the given speed is invariant. For the light speed such transformations are the Lorentz transformations and Einstein's law of the velocity compositions.

However, the light speed is not invariant, and its value does not restrict the values of other speeds, in the Galilean - Newtonian spaces. This confirms the results
of Goy [9, p. 98], Keswany [27, pp. 135, 138], Martin [47, pp. 53, 54], and Wesley [52, p. 261]. This does not contradict the famous experimental results of Fizeau, Michelson and Morley, because they do not prove the light speed invariance, which is shown in the book [25].

In this connection see also the works of Ceapa [1], [2], Kracklauer [28], Marmet [44], [46], Martin [47], and Wesley [52]. They all verify the obvious fact that the relative light speeds are different with respect to a railway station, and with respect to a fast moving train, with respect to a flying rocket et, and with respect to another light signal, all moving in parallel and in the same sense in the Galilean - Newtonian space.

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