Progress Toward a Classical Theory of Matter

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1. Introduction

An understanding of the classical properties of light was developed by 19th century scientists on the basis of Thomas Young's suggestion in 1817 that light waves consist of transverse vibrations such as occur in an elastic solid. For a comprehensive history, see Whittaker [1951]. There have also been recent attempts to revive this model of the aether [Dmitriyev 1992, Hatch 1992, Karlsen 1998]. According to this model light consists of transverse waves whose evolution is described by a second order differential equation. Actually, the appropriate wave equation for ideal elastic waves has not been conclusively established. Wave equations have been derived for an ideal elastic solid from analysis of stress and strain, but there is some confusion as to how the theory should accommodate rotations. For instance, in Kleinert (1989) additional elastic constants were introduced *ad hoc* for this purpose.

The wave nature of matter was first proposed by de Broglie [1924] and subsequently confirmed in experiments by Davisson and Germer [1927], and independently by Thomson and Reid [1927]. However the equations developed to describe these 'matter waves' are first order equations rather than second-order wave equations. Although these waves are commonly interpreted as probability waves, the quantum mechanical Dirac equation is also a deterministic equation for the evolution of angular momentum density and other physical observables. As such, it should correspond to classical wave theory. Others have reformulated the Dirac theory in terms of deterministic relations between local physical observables [Takabayashi 1957, Hestenes 1973]. However these investigators did not construct a corresponding classical wave theory.

Several attempts have been made to describe elementary particles as soliton (or particle-like) wave solutions of a nonlinear Dirac equation. See Ra□ada [1983] for a short review and Gu [1998] for a more recent discussion of this approach to understanding matter. The soliton solutions found to date do not appear to correspond to matter, although some similarities have been claimed. It is not clear how Dirac solitons relate to ordinary classical wave solitons.

In this paper we focus on analyzing the solutions to simple wave equations. We find that the bispinors which are associated with matter in quantum mechanics are in fact the general solutions of ordinary scalar or vector wave equations. However, the mathematical structure of classical wave bispinors differs significantly from that of Dirac's bispinors. Unlike the Dirac algebra, classical wave bispinors can be factored with independent rotations of velocity and polarization. We subsequently discuss how these solutions may be applicable to the equations of waves in an ideal elastic solid, and how they relate to the electron equation, electromagnetic potentials, and the parity transformation. We also describe briefly how classical wave theory can explanation various physical phenomena such as special relativity and gravity.

2. One-Dimensional Scalar Waves

Consider a scalar quantity (a) which satisfies a wave equation with wave speed (c) in one spatial dimension (z):

$$\frac{\partial^2}{\partial t^2} a = c^2 \frac{\partial^2}{\partial z^2} a \qquad (1)$$

This equation can be factored:

$$\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right]\left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right] a = 0 \qquad (2)$$

The solution for each operator is an arbitrary function of $(z \pm ct)$, with the minus sign corresponding to a wave propagating forward (a_F) and the plus sign corresponding to a wave propagating backward (a_B):

$$\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right] a_F(z - ct) = 0 \quad ; \quad \left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right] a_B(z + ct) = 0 \tag{3}$$

If we write these equations in matrix form:

$$\left[\frac{\partial}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c\frac{\partial}{\partial z}\right] \begin{pmatrix} a_F(z - ct) \\ a_B(z + ct) \end{pmatrix} = 0 \tag{4}$$

then we have reduced the second-order wave equation to a first-order matrix equation.

The general solution is a superposition of forward and backward propagating waves:

$$a(z,t) = a_F(z - ct) + a_B(z + ct) \tag{5}$$

This solution to the one-dimensional wave equation can be found in any elementary textbook on waves.

*2.1. **Physical Observables***

Suppose we wish to determine the wave parameters from measurements. At each point in space and time, we need only determine two real parameters: a_F and a_B. The two parameters $\partial a/\partial t$ and $\partial a/\partial z$ are sufficient to determine the derivatives:

$$\begin{aligned} \partial a_F/\partial t &= -c\partial a_F/\partial z = [\partial a/\partial t - c\,\partial a/\partial z]/2 \\ \partial a_B/\partial t &= c\,\partial a_B/\partial z = [\partial a/\partial t + c\,\partial a/\partial z]/2 \end{aligned} \tag{6}$$

Furthermore a is equal to the sum $a_F + a_B$. However, the difference $a_F - a_B$, and hence the separate values of a_F and a_B, cannot be determined from measurements of a. The constant component of the difference $a_F - a_B$ is in fact completely arbitrary. Because we can only determine the wave derivatives and not the actual amplitudes of the components, we should only work with the derivatives. The forward and backward components are thus defined by the above equation. The general solution is then:

$$a(z,t) = a(z,t_0) + \int_{t_0}^{t} d\tau \left[\dot{a}_F(z,\tau) + \dot{a}_B(z,\tau)\right] \tag{7}$$

Let $\dot{a} \equiv \partial a/\partial t$ and $a' \equiv \partial a/\partial z$. The forward and backward components can be combined to yield:

$$\dot{a} = \dot{a}_F + \dot{a}_B \quad ; \quad a' = a'_F + a'_B \tag{8}$$

*2.2. **Spinors and Bispinors***

If we regard the z-axis as one of three orthogonal axes, then the two independent components $\dot{a}_F$ and $\dot{a}_B$ differ by a 180 degree rotation. This is the definitive property of independent states in spin one-half systems. Unfortunately, this property is de-emphasized in the physics literature in favor of the more exotic property that complex spinors change sign upon 360 degree rotation. This latter property does not apply to physical observables which are computed from bilinear products of spinors. However, the separation of independent states by 180 degrees does apply to wave velocity, implying that solutions of the wave equation generally form spin one-half systems. Note that unlike positive and negative scalars or vector components (which can also be expressed as bilinear products of spinors), waves with positive and negative velocity are not related by a multiplicative factor of minus one. Superposition of waves with positive and negative velocities does not merely form a single wave with some intermediate velocity (although it is possible to compute an average velocity). The forward and backward waves are independent states. The mathematical explanation of this property is that wave velocity is a property of the functional arguments and is not simply an amplitude.

The relationship between waves and spinors can be made explicit as in Close [2002] by further decomposition into positive-definite components ($\dot{a}_{\mathrm{F+}}, \dot{a}_{\mathrm{B+}}, \dot{a}_{\mathrm{F-}}, \dot{a}_{\mathrm{B-}}$) or ($a'_{\mathrm{F+}}, a'_{\mathrm{B+}}, a'_{\mathrm{F-}}, a'_{\mathrm{B-}}$) representing positive (+) or negative (–) contributions to the wave derivatives:

$$\begin{aligned} \dot{a}(z,t) &= \dot{a}_{\mathrm{F+}}(z-ct) - \dot{a}_{\mathrm{F-}}(z-ct) + \dot{a}_{\mathrm{B+}}(z+ct) - \dot{a}_{\mathrm{B-}}(z+ct) \\ ca'(z,t) &= -\dot{a}_{\mathrm{F+}}(z-ct) + \dot{a}_{\mathrm{F-}}(z-ct) + \dot{a}_{\mathrm{B+}}(z+ct) - \dot{a}_{\mathrm{B-}}(z+ct) \end{aligned} \tag{9}$$

From here on the functional arguments will not be written explicitly. Since each component has a unique sign, we can express $\dot{a}$ and a' in spinorial form with the one-dimensional wave function ψ_{v} (the subscript 'v' refers to the velocity axis):

$$\dot{a} = \begin{pmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv \psi_{\mathrm{v}}^T \sigma \psi_{\mathrm{v}}$$

$$ca' = \begin{pmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv -\psi_{\mathrm{v}}^T \beta_{\mathrm{v}} \sigma \psi_{\mathrm{v}} \tag{10}$$

where the superscript T indicates transposition of the column matrix and the matrix β_{v} tabulates the forward and backward velocities (v):

$$v\psi_{\mathrm{v}} = c \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv c\beta_{\mathrm{v}}\psi_{\mathrm{v}} \tag{11}$$

where $\psi_{\mathrm{v}} = \begin{bmatrix} \dot{a}_{\mathrm{B+}}^{1/2} & \dot{a}_{\mathrm{B-}}^{1/2} & \dot{a}_{\mathrm{F+}}^{1/2} & \dot{a}_{\mathrm{F-}}^{1/2} \end{bmatrix}^T$.

This wave function is a one-dimensional bispinor. Each pair of terms ($\dot{a}_{\mathrm{B+}}^{1/2}, \dot{a}_{\mathrm{F+}}^{1/2}$) and ($\dot{a}_{\mathrm{B-}}^{1/2}, \dot{a}_{\mathrm{F-}}^{1/2}$) constitutes a one-dimensional spinor. In one dimension the components of the bispinor may be taken to be real and positive-definite. We will see later that extension to three dimensions requires complex components.

Changing the order of terms in the wave function is called a change of 'representation'. A few important points are:

1. The components of the column matrix wave function are real and positive-definite.
2. Only one forward component and one backward component can be non-zero at any given time and place.
3. The spatio-temporal variation of each component must be consistent with its location in the column matrix.

Since some of the components must be zero, let δ_{F} and δ_{B} be either zero or one. Then:

$$\psi_{\mathrm{v}} = \begin{bmatrix} \dot{a}_{\mathrm{F}}^{1/2}\delta_{\mathrm{F}} & \dot{a}_{\mathrm{F}}^{1/2}[1-\delta_{\mathrm{F}}] & \dot{a}_{\mathrm{B}}^{1/2}\delta_B & \dot{a}_{\mathrm{B}}^{1/2}[1-\delta_B] \end{bmatrix}^T \tag{12}$$

The equation of evolution of the wave components is:

$$\frac{\partial}{\partial t}\psi_{\mathrm{v}} + c\beta_{\mathrm{v}}\frac{\partial}{\partial z}\psi_{\mathrm{v}} = 0 \tag{13}$$

This equation can be interpreted as a convective derivative ($\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + c\beta_\mathrm{v}\,\partial/\partial z$) with two opposite velocities represented by the matrix $v=c\beta_\mathrm{v}$.

The relation between one dimensional bispinor equations and scalar wave equations can be summarized as follows:

Bispinor	Scalar
$\frac{\partial}{\partial t}\left[\psi_\mathrm{v}^T \sigma \psi_\mathrm{v}\right] + c\frac{\partial}{\partial z}\left[\psi_\mathrm{v}^T \beta_\mathrm{v} \sigma \psi_\mathrm{v}\right] = 0$	$\frac{\partial^2}{\partial t^2} a - c^2 \frac{\partial^2}{\partial z^2} a = 0$
$\frac{\partial}{\partial t}\left[\psi_\mathrm{v}^T \psi_\mathrm{v}\right] + c\frac{\partial}{\partial z}\left[\psi_\mathrm{v}^T \beta_\mathrm{v} \psi_\mathrm{v}\right] = 0$	$\frac{\partial}{\partial t}\left\|\frac{\partial a_F}{\partial t}\right\| + \frac{\partial}{\partial t}\left\|\frac{\partial a_B}{\partial t}\right\| + c^2 \frac{\partial}{\partial z}\left\|\frac{\partial a_F}{\partial z}\right\| - c^2 \frac{\partial}{\partial z}\left\|\frac{\partial a_B}{\partial z}\right\| = 0$
$\frac{\partial}{\partial t}\left[\psi_\mathrm{v}^T \beta_\mathrm{v} \sigma \psi_\mathrm{v}\right] + c\frac{\partial}{\partial z}\left[\psi_\mathrm{v}^T \sigma \psi_\mathrm{v}\right] = 0$	$\frac{\partial}{\partial t}\left[-c\frac{\partial a}{\partial z}\right] + c\frac{\partial}{\partial z}\left[\frac{\partial a}{\partial t}\right] = 0$
$\frac{\partial}{\partial t}\left[\psi_\mathrm{v}^T \beta_\mathrm{v} \psi_\mathrm{v}\right] + c\frac{\partial}{\partial z}\left[\psi_\mathrm{v}^T \psi_\mathrm{v}\right] = 0$	$c\frac{\partial}{\partial t}\left\|\frac{\partial a_F}{\partial z}\right\| - c\frac{\partial}{\partial t}\left\|\frac{\partial a_B}{\partial z}\right\| + c\frac{\partial}{\partial z}\left\|\frac{\partial a_F}{\partial t}\right\| + c\frac{\partial}{\partial z}\left\|\frac{\partial a_B}{\partial t}\right\| = 0$

(14)

*2.3. **Wave Velocity***

The mean velocity (v) of the wave is proportional to the ratio between the difference and sum of the forward and backward components [Close 2002]:

$$v = c\frac{|\dot{a}_F| - |\dot{a}_B|}{|\dot{a}_F| + |\dot{a}_B|} = c\frac{\psi_\mathrm{v}^T \beta_\mathrm{v} \psi_\mathrm{v}}{\psi_\mathrm{v}^T \psi_\mathrm{v}} \tag{15}$$

Since $|\dot{a}_F|$ and $|\dot{a}_B|$ are positive-definite, we can define them by the relation:

$$\begin{aligned} |\dot{a}_F| &= \dot{a}_0 \exp(\alpha) \\ |\dot{a}_B| &= \dot{a}_0 \exp(-\alpha) \end{aligned} \tag{16}$$

so that our definition of velocity is:

$$v = c\frac{\dot{a}_0 \exp(\alpha) - \dot{a}_0 \exp(-\alpha)}{\dot{a}_0 \exp(\alpha) + \dot{a}_0 \exp(-\alpha)} = c \tanh\alpha \tag{17}$$

If we start from a zero-velocity state with $|\dot{a}_\mathrm{F}| = |\dot{a}_\mathrm{B}| = \dot{a}_0$, then we can change the velocity using the 'Lorentz boost' operator ($\psi_\mathrm{v} \to \exp(\beta_\mathrm{v}\,\alpha/2)\psi_\mathrm{v}$):

$$v = \frac{\left[\psi_\mathrm{v}^T \exp(\beta_\mathrm{v}\,\alpha/2)\right] c\beta_\mathrm{v} \left[\exp(\beta_\mathrm{v}\,\alpha/2)\psi_\mathrm{v}\right]}{\left[\psi_\mathrm{v}^T \exp(\beta_\mathrm{v}\,\alpha/2)\right]\left[\exp(\beta_\mathrm{v}\,\alpha/2)\psi_\mathrm{v}\right]} = c\frac{\exp(\alpha) - \exp(-\alpha)}{\exp(\alpha) + \exp(-\alpha)} = c\tanh\alpha \tag{18}$$

Using Lorentz boosts, the wave function can be written as:

$$\psi_\mathrm{v} = \frac{\dot{a}_0^{1/2}}{\sqrt{2}} \exp(\beta_\mathrm{v}\,\alpha/2)\left[\delta_\mathrm{F} \quad [1-\delta_\mathrm{F}] \quad \delta_B \quad [1-\delta_B]\right]^T \tag{19}$$

This form has two independent continuous parameters and two binary parameters.

3. Three Dimensional Scalar Waves

3.1. *Rotation of Gradient and Velocity*

The spatial derivative $\partial/\partial z$ generalizes in three dimensions to a arbitrary direction $\partial/\partial x_{\mathrm{v}}$. Since the matrix β_{v} is associated with a particular axis (parallel to gradient), it must be one component of a vector. We can let $\beta_{\mathrm{v}} \equiv \beta_3$ and define the other velocity matrix components as:

$$\beta_1 = \tilde{1}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \tilde{1}\begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad \beta_3 = \tilde{1}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

We must now allow the wave function to have complex components. The unit pseudoscalar is denoted by $\tilde{1}$, and the overhead tilde will generally be used to denote a pseudoscalar. Note that these matrices have commutation relations similar to the Pauli matrices (σ_i^P):

$$\beta_i\beta_j + \beta_j\beta_i = 2\delta_{ij} \; ; \quad \beta_i\beta_j - \beta_j\beta_i = 2\,\tilde{\mathrm{i}}\,\varepsilon_{ijk}\beta_k \tag{21}$$

The rotation operators for this space have the form:

$$R_{\beta_j}(\zeta_i) = \exp\left(-\tilde{\mathrm{i}}\,\beta_i\zeta_i/2\right)\beta_j \exp\left(\tilde{\mathrm{i}}\,\beta_i\zeta_i/2\right) = \beta_j\cos\zeta_i - \frac{\tilde{\mathrm{i}}}{2}\left[\beta_i\beta_j - \beta_j\beta_i\right]\sin\zeta_i \tag{22}$$

For instance, rotation of β_2 by $\pi/2$ about β_1 yields β_3.

3.2. *Wave Function*

In three dimensions the gradient can be defined as a one-dimensional derivative rotated by angle $\boldsymbol{\zeta}$ to a new axis $\hat{\mathbf{v}}$. Let:

$$\begin{aligned} \beta_{\mathrm{v}} &\equiv \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\beta_3 \exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right) \\ \psi &\equiv \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_{\mathrm{v}} \end{aligned} \tag{23}$$

Note that rotations are defined relative to a default orientation of velocity along the x_3 axis. The three-dimensional gradient is:

$$\nabla a = R_{\zeta_i}\left\{\hat{\mathbf{x}}_3 \frac{\partial a}{\partial x_3}\right\} = -\hat{\mathbf{v}}\psi_{\mathrm{v}}^T c\beta_3\sigma\psi_{\mathrm{v}} = -c\psi^{\dagger}\boldsymbol{\beta}\sigma\psi \tag{24}$$

Note that rotation changes the direction but not the numerical value of the derivative. The exponential factors applied to the velocity matrix cancel the exponential factors applied to the wave functions. Writing a column matrix as the transpose of a row matrix, the rotated wave function ψ can be written as:

$$\psi = \frac{\dot{a}_0^{1/2}}{\sqrt{2}}\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\exp\left(\beta_3\,\alpha/2\right)\left[\delta_{\mathrm{F}} \quad [1-\delta_{\mathrm{F}}] \quad \delta_B \quad [1-\delta_B]\right]^T \tag{25}$$

However, in three dimensions the constant column matrix which represents v_3=0 states may have nonzero velocity perpendicular to x_3. This is indeed the case for $\psi_0 = [1 \;\; 0 \;\; 1 \;\; 0]^T$ and $\psi_0 = [0 \;\; 1 \;\; 0 \;\; 1]^T$. The remaining states with zero velocity are obtained by rotation of velocity from:

$$\psi_0 = [1 \;\; 0 \;\; 0 \;\; 1]^T \tag{26}$$

This state has zero time derivative but nonzero gradient. When Lorentz boosts are applied both the time derivative and velocity can be non-zero. The final form of the wave function is thus:

$$\psi = \frac{\dot{a}_0^{1/2}}{\sqrt{2}} \exp\left(-\tilde{\mathrm{i}}\, \boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\exp\left(\beta_3\, \alpha/2\right)\psi_0 = \frac{\dot{a}_0^{1/2}}{\sqrt{2}} \exp\left(\boldsymbol{\beta}\cdot\boldsymbol{\alpha}/2\right)\exp\left(-\tilde{\mathrm{i}}\, \boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_0 \tag{27}$$

This is the general form of the scalar wave function. In the final expression the angle $\boldsymbol{\zeta}$ is the angle between the x_3 axis and the velocity parameter $\boldsymbol{\alpha}$. The constant matrix is multiplied by factors representing an amplitude, a 1-D velocity boost, and a general rotation in velocity space (two angles to determine velocity direction plus rotation about the velocity axis). Clearly four parameters are needed to determine $\partial a/\partial t$ and ∇a. The significance of rotation about the velocity axis, will be discussed in the next section.

*3.3. **First-Order Wave Equation***

The time derivative of (27) yields the first-order wave equation:

$$\frac{\partial}{\partial t}\psi = -\frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi + \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\frac{\partial}{\partial t}\left[\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi\right] \tag{28}$$

Here we can see the effect of rotation about the velocity axis. Rotation of the left-hand side involves only direct rotation of the wave function, but rotation of the right-hand side also involves rotation of the angular frequency $\partial\boldsymbol{\zeta}/\partial t$.

The time derivative of the one-dimensional wave function can be replaced by a spatial derivative:

$$\frac{\partial}{\partial t}\psi = -\frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)c\beta_3\hat{\mathbf{v}}\cdot\nabla\left[\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi\right] \tag{29}$$

Substituting $\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\beta_3\hat{\mathbf{v}}\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right) = \boldsymbol{\beta}$ into (29) yields:

$$\frac{\partial}{\partial t}\psi = -\frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\boldsymbol{\beta}\cdot\nabla\psi - c\frac{\tilde{\mathrm{i}}}{2}\nabla\cdot\boldsymbol{\zeta}\psi + \frac{c}{2}\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\boldsymbol{\beta}\psi \tag{30}$$

The equation of evolution of the scalar wave amplitude is:

$$\frac{\partial}{\partial t}\left[\psi^{\dagger}\sigma\psi\right] = -\nabla\cdot\psi^{\dagger}c\boldsymbol{\beta}\sigma\psi + c\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\left[\psi^{\dagger}\sigma\boldsymbol{\beta}\psi\right] \tag{31}$$

Which, in terms of polarization is:

$$\frac{\partial^2}{\partial t^2}a = c^2\nabla^2 a - c^2\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\nabla a \tag{32}$$

The extra term is inherently three dimensional, but it is not clear whether it has some physical basis or is simply spurious.

4. Vector Waves

Next we consider vector waves (polar or axial vectors). Each component of the polarization vector can be regarded as a scalar (true scalar or pseudo-scalar). Therefore it should be possible to describe an arbitrary polarization vector by a scalar and three rotation angles. Since scalar waves require five parameters, we expect vector waves to require eight parameters. As with velocity rotations, only two angles are necessary to determine the direction of polarization, but a third angle is necessary in order to fully describe changes in the polarization direction.

4.1. Rotation of Polarization

Recall that the scalar polarization is $\dot{a} = \psi^T \sigma \psi$. We now regard this as one component of a vector: $\dot{a}_3 = \psi^T \sigma_3 \psi$. The vector a could be polar or axial, but we will assume an axial vector (pseudovector). Each component of a vector changes sign upon rotation by π radians and the two-component spinor pairs are ($\dot{a}_{\mathrm{F}+}, \dot{a}_{\mathrm{B}-}$) and ($\dot{a}_{\mathrm{F}-}, \dot{a}_{\mathrm{B}+}$). The three orthogonal polarization matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{33}$$

These matrices have the same commutation relations as the Pauli matrices (σ_i^P) and the velocity matrices (β_i):

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \;;\; \sigma_i \sigma_j - \sigma_j \sigma_i = 2\,\mathrm{i}\,\varepsilon_{ijk} \sigma_k \tag{34}$$

The rotation operators for this space are similar to the velocity matrix rotation operators:

$$R_{\sigma_j}(\xi_i) = \exp(-\mathrm{i}\,\sigma_i \xi_i/2)\sigma_j \exp(\mathrm{i}\,\sigma_i \xi_i/2) = \sigma_j \cos\xi_i - \frac{\mathrm{i}}{2}\left[\sigma_i \sigma_j - \sigma_j \sigma_i\right]\sin\xi_i \tag{35}$$

4.2. Factorization and First-Order Wave Equation

Including rotation of velocity by angle $\boldsymbol{\zeta}$ and polarization axis by angle $\boldsymbol{\xi}$ yields:

$$\psi = \frac{\dot{a}_0^{1/2}}{\sqrt{2}} \exp(-\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2)\exp(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2)\exp(\beta_3\,\alpha/2)\psi_0 \tag{36}$$

The wave function has eight free parameters: an amplitude, three polarization angles, three velocity angles, and a boost along the velocity direction.

Using the same methods as for scalar waves, the first-order wave equation is found to be:

$$\frac{\partial}{\partial t}\psi + c\boldsymbol{\beta}\cdot\nabla\psi = -\frac{\partial}{\partial t}\left[\boldsymbol{\xi}\cdot\frac{\mathrm{i}\,\boldsymbol{\sigma}}{2} + \boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\right]\psi - c\boldsymbol{\beta}\cdot\nabla\left[\boldsymbol{\xi}\cdot\frac{\mathrm{i}\,\boldsymbol{\sigma}}{2} + \boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\right]\psi \tag{37}$$

Expanding:

$$\frac{\partial}{\partial t}\psi + c\boldsymbol{\beta}\cdot\nabla\psi = \frac{1}{2}\left[-\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\mathrm{i}\,\boldsymbol{\sigma} - c[\boldsymbol{\beta}\cdot\nabla\boldsymbol{\xi}]\cdot\mathrm{i}\,\boldsymbol{\sigma} - \frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\tilde{\mathrm{i}}\,\boldsymbol{\beta} - \tilde{\mathrm{i}}\,c\nabla\cdot\boldsymbol{\zeta} + c[\nabla\times\boldsymbol{\zeta}]\cdot\boldsymbol{\beta}\right]\psi \tag{38}$$

4.3. Physical observables

Derivatives of the wave amplitude are defined in terms of classical bispinors as:

$$\frac{\partial}{\partial t}a_j = \psi^\dagger \sigma_j \psi \;;\; c\partial_i a_j = -\psi^\dagger \beta_i \sigma_j \psi \tag{39}$$

Note that the constant wave function ψ_0 yields a non-zero component:

$$c\partial_1 a_1 = \psi_0^\dagger \beta_1 \sigma_1 \psi_0 = 1 \tag{40}$$

Altogether we have defined 16 physical observables associated with the following matrices: the identity *I*, three velocity components β_i, three polarization components σ_j, and nine components of the gradient of polarization $\beta_i\sigma_j$. The factorization of the wave function includes a single amplitude, rotations of polarization and velocity, and velocity boosts for a total of eight independent factors which completely define the four-element complex column matrix wave function for a given polarization parity.

4.4. Vector wave equation

Start from (37), multiply $\psi^\dagger \boldsymbol{\sigma}$ and add the transpose equation to obtain the time derivative of the polarization:

$$\frac{\partial}{\partial t}\left[\psi^\dagger \boldsymbol{\sigma}\psi\right] = -c\partial_i\left[\psi^\dagger \beta_i \boldsymbol{\sigma}\psi\right] + \frac{\partial \boldsymbol{\xi}}{\partial t}\times\psi^\dagger\boldsymbol{\sigma}\psi + \frac{\partial \boldsymbol{\xi}}{\partial x_i}\times\psi^\dagger\beta_i\boldsymbol{\sigma}\psi + c\left[\nabla\times\boldsymbol{\zeta}\right]_i\psi^\dagger\beta_i\boldsymbol{\sigma}\psi \tag{41}$$

In terms of the vector polarization ($\mathbf{a}$) this wave equation is:

$$\frac{\partial^2}{\partial t^2}\mathbf{a} = c^2\nabla^2\mathbf{a} + \frac{\partial \boldsymbol{\xi}}{\partial t}\times\dot{\mathbf{a}} - c^2\frac{\partial \boldsymbol{\xi}}{\partial x_i}\times\frac{\partial \mathbf{a}}{\partial x_i} - c^2\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\nabla\mathbf{a} \tag{42}$$

The angles $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ represent rotations of polarization direction $\hat{\mathbf{a}}$ and gradient direction $\hat{\mathbf{v}}$, respectively. Therefore:

$$\frac{\partial^2}{\partial t^2}\mathbf{a} - \dot{a}\frac{\partial}{\partial t}\hat{\mathbf{a}} = c^2\nabla^2\mathbf{a} - c^2\left[\nabla a\right]\cdot\nabla\hat{\mathbf{a}} - c^2\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\nabla\mathbf{a} \tag{43}$$

4.5. Comparison with Dirac Wave Functions

The chiral representation of the Dirac matrices in our notation (ignoring factors of the unit pseudoscalar $\tilde{1}$) is:

$$\begin{array}{ll} \gamma^0 = \beta_1 & \gamma^i = \gamma^0\gamma^5\sigma^i \quad (i = 1,2,3) \\ \mathrm{i}\,\gamma^0\gamma^5 = \beta_2 & \hat{\mathrm{i}} = \mathrm{i}\,\gamma^5 = \mathrm{i}\,\beta_3 = -\gamma^0\gamma^1\gamma^2\gamma^3 \end{array} \tag{44}$$

The Dirac wave function can be factorized as (see e.g. Hestenes [1967]):

$$\psi = \frac{A^{1/2}}{\sqrt{2}}\exp\left(-\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(\beta_3\,\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2\right)\exp\left(-\mathrm{i}\,\beta_3\,\zeta_3/2\right)\psi_1 \tag{45}$$

This factorization is not consistent with the classical bispinor factorization because there is no factor representing rotation of velocity $\psi^\dagger\beta_3\boldsymbol{\sigma}\psi$ independently from polarization $\psi^\dagger\boldsymbol{\sigma}\psi$. One can rotate the velocity parameter $\boldsymbol{\alpha}$, but no such rotation operator appears in the factorization.

4.6. Classical Electron Equation

Suppose that for an undisturbed wave the coefficient on the right side of (37) is a matrix term of the form $-\mathrm{i}\,\mathbf{M}\cdot\boldsymbol{\mu}$:

$$\frac{\partial}{\partial t}\psi + c\boldsymbol{\beta}\cdot\nabla\psi(\mathbf{x},t) = -\mathrm{i}\,\mathbf{M}\cdot\boldsymbol{\mu}\psi \tag{46}$$

This is the classical analogue of the free electron equation. The mass term is associated with rotation of velocity and perhaps also polarization. Rotation of the wave function by π radians about any axis perpendicular to $\boldsymbol{\mu}$ changes the sign of the mass term. The change in sign of mass is commonly associated with transformation between matter and anti-matter [Hestenes 1990]. Rotation by π about the velocity axis changes the orbits from clockwise to counterclockwise, or vice-versa.

4.7. Classical Bispinor Potentials

If the evolution of the velocity (or gradient) angle is subsequently perturbed ($\boldsymbol{\zeta} \to \boldsymbol{\zeta} + \delta\boldsymbol{\zeta}$) then the perturbed equation is:

$$\frac{\partial}{\partial t}\psi + c\boldsymbol{\beta}\cdot\nabla\psi = -\mathrm{i}\mathbf{M}\cdot\boldsymbol{\mu}\psi - \frac{\tilde{\mathrm{i}}}{2}\boldsymbol{\beta}\cdot\frac{\partial}{\partial t}\delta\boldsymbol{\zeta} - \frac{\tilde{\mathrm{i}}\,c}{2}\nabla\cdot\delta\boldsymbol{\zeta} + \frac{c}{2}\boldsymbol{\beta}\cdot[\nabla\times\delta\boldsymbol{\zeta}]\psi \tag{47}$$

Regrouping:

$$\left[\frac{\partial}{\partial t} + \frac{\tilde{\mathrm{i}}}{2}c\nabla\cdot\delta\boldsymbol{\zeta}\right]\psi + c\boldsymbol{\beta}\cdot\left[\nabla + \frac{\tilde{\mathrm{i}}}{2}\frac{1}{c}\frac{\partial}{\partial t}\delta\boldsymbol{\zeta}\right]\psi = -\mathrm{i}\,\mathbf{M}\cdot\boldsymbol{\mu}\psi - \frac{c}{2}\boldsymbol{\beta}\cdot[\nabla\times\delta\boldsymbol{\zeta}]\psi \tag{48}$$

Define the classical bispinor potentials as:

$$eV \equiv \frac{\tilde{1}}{2}c\nabla\cdot\delta\boldsymbol{\zeta} \;\;;\;\; e\mathbf{A} \equiv -\frac{\tilde{1}}{2}\frac{\partial}{\partial t}\delta\boldsymbol{\zeta} \tag{49}$$

The perturbed equation is:

$$\left[\frac{\partial}{\partial t} + \mathrm{i}\,eV\right]\psi + c\boldsymbol{\beta}\cdot\left[\nabla - \mathrm{i}\frac{e}{c}\mathbf{A}\right]\psi = -\mathrm{i}\,\mathbf{M}\cdot\boldsymbol{\mu}\psi - \frac{c}{2}\boldsymbol{\beta}\cdot[\nabla\times\delta\boldsymbol{\zeta}]\psi \tag{50}$$

This is the form in which electromagnetic potentials enter the electron equation (neglecting the final term in the above equation). Note that if e is taken as constant then the Lorenz condition is automatically satisfied:

$$\frac{1}{c}\frac{\partial}{\partial t}V + \nabla\cdot\mathbf{A} \equiv 0 \tag{51}$$

Hence the basic form of the electromagnetic potentials emerges naturally from perturbation of the velocity angle of the classical bispinor wave function. No such potentials can be derived from the usual factorization of quantum mechanical wave functions. The potentials represent perturbations of the wave velocity rotation. Potentials defined in this way describe the perturbed motion of the wave without explicit reference to any specific cause of the perturbation.

4.8. Hamiltonian Formulation

Hamilton's equations of motion have the form (e.g. Morse and Feshbach [1953a]):

$$\frac{\partial\psi}{\partial t} = \frac{\partial \mathrm{H}}{\partial p_\psi} \tag{52}$$

where ψ is a field variable and p_ψ is the conjugate 'momentum' to the field defined by:

$$p_\psi = \frac{\partial \mathrm{H}}{\partial[\partial\psi/\partial t]} \tag{53}$$

We can fit the bispinor equation to this form by defining momenta conjugate to the wave functions:

$$p_\psi = \frac{\mathrm{i}}{2}\psi^\dagger \tag{54}$$

and the Hamiltonian:

$$\mathrm{H} = \frac{\mathrm{i}}{2}\left[\psi^\dagger\frac{\partial\psi}{\partial t} - \frac{\partial\psi^\dagger}{\partial t}\psi\right] = \frac{1}{2}\psi^\dagger\left\{-c\boldsymbol{\beta}\cdot\mathrm{i}\nabla + \frac{1}{2}\left[\frac{\partial}{\partial t} + c\boldsymbol{\beta}\cdot\nabla\right][\boldsymbol{\xi}\cdot\boldsymbol{\sigma} + \tilde{1}\boldsymbol{\zeta}\cdot\boldsymbol{\beta}]\right\}\psi + c.c. \tag{55}$$

From here on it will simply be assumed that the imaginary part is discarded. Notice that the Hamiltonian H will have units of energy if the wave polarization has units of angular momentum. Substituting the mass and electromagnetic potentials yields:

$$\mathrm{H} = \mathrm{i}\psi^\dagger \frac{\partial \psi}{\partial t} = \psi^\dagger \left\{ -c\boldsymbol{\beta}\cdot \mathrm{i}\nabla\psi + \mathbf{M}\cdot\boldsymbol{\mu}\psi - \boldsymbol{\beta}\cdot\frac{e}{c}\mathbf{A} + eV - \frac{c}{2}\boldsymbol{\beta}\cdot[\mathrm{i}\nabla\times\delta\boldsymbol{\zeta}] \right\}\psi \tag{56}$$

We can also define a Hamiltonian operator with $\frac{\partial \psi}{\partial t} = -\mathrm{i}\,H\psi$:

$$\mathrm{H} \equiv \psi^\dagger H\psi = \psi^\dagger \left\{ -c\boldsymbol{\beta}\cdot \mathrm{i}\nabla\psi + \mathbf{M}\cdot\boldsymbol{\mu}\psi - \boldsymbol{\beta}\cdot\frac{e}{c}\mathbf{A} + eV - \frac{c}{2}\boldsymbol{\beta}\cdot[\mathrm{i}\nabla\times\delta\boldsymbol{\zeta}] \right\}\psi \tag{57}$$

Since $\boldsymbol{\beta}$ is a velocity operator, the conjugate momentum for $\mathbf{r}$ is:

$$\mathbf{p_r} = \frac{\partial \mathrm{H}}{\partial[c\boldsymbol{\beta}]} = \psi^\dagger \left\{ -\mathrm{i}\nabla - \frac{e}{c}\mathbf{A} + \frac{1}{2}\nabla[\boldsymbol{\xi}\cdot\boldsymbol{\sigma}] - \frac{\mathrm{i}}{2}\nabla\times\boldsymbol{\zeta} \right\}\psi \tag{58}$$

The conjugate momentum for rotation has contributions from the angular directions of the velocity (β_θ,β_ϕ) and from the time derivatives of rotation angles (ξ,ζ) to yield:

$$\mathbf{p_\varphi} = \frac{\partial \mathrm{H}}{\partial[\partial\boldsymbol{\varphi}/\partial t]} = \frac{\partial \mathrm{H}}{\partial\left[\hat{\mathbf{r}}\times\frac{c\boldsymbol{\beta}}{r}\right]} + \frac{\partial \mathrm{H}}{\partial[\partial\boldsymbol{\xi}/\partial t]} + \frac{\partial \mathrm{H}}{\partial[\partial\boldsymbol{\zeta}/\partial t]} = \psi^\dagger \left\{ -\mathrm{i}\,\mathbf{r}\times\nabla + \frac{1}{2}[\boldsymbol{\sigma}+\boldsymbol{\beta}] \right\}\psi \tag{59}$$

The conjugate momentum for time is simply H itself:

$$\mathbf{p}_t = \mathrm{H} \tag{60}$$

The time derivative of any observable Q is:

$$\frac{\partial}{\partial t}\left[\psi^\dagger Q\psi\right] = \left[\frac{\partial}{\partial t}\psi^\dagger\right]Q\psi + \psi^\dagger Q\left[\frac{\partial}{\partial t}\right]\psi + \psi^\dagger\frac{\partial Q}{\partial t}\psi = \psi^\dagger\,\mathrm{i}[Q,H]\psi + \psi^\dagger\frac{\partial Q}{\partial t}\psi \tag{61}$$

For example, keeping only the electromagnetic terms yields:

$$\frac{\partial \mathbf{p_r}}{\partial t} = \psi^\dagger \left\{ c\boldsymbol{\beta}\times\frac{e}{c}\mathbf{B} + e\mathbf{E} \right\}\psi \tag{62}$$

Hence the Lorenz force has a simple interpretation in terms of derivatives of wave velocity rotation. The classical equation above differs from the quantum mechanical equation only in the matrix form of the velocity operator.

4.9. Parity

The quantum mechanical matrices (or unit vectors) γ^0, i $\gamma^0\gamma^5$, and γ^5 which correspond to $\boldsymbol{\beta}$ represent wave velocity for classical wave functions. Under spatial inversion all of the directions associated with velocity (or gradient) must be reversed. The right-handed coordinate basis (γ^0, i $\gamma^0\gamma^5$, γ^5) becomes the left-handed coordinate basis ($-\gamma^0$, $-$i $\gamma^0\gamma^5$, $-\gamma^5$). This property also applies to the quantum mechanical wave function, for which these γ matrices may be interpreted as directions relative to the velocity. The conventional parity operator is $P\psi(-\mathbf{r})=\gamma^0\psi(\mathbf{r})$, but since this does not invert $\psi^\dagger\gamma_0\psi$ it is classically incorrect. This operator is a rotation in $\boldsymbol{\beta}$-space. The conventional derivation of the parity operator is questionable because of the arbitrary assumption that the matrix γ^0 does not change sign under spatial inversion.

It is impossible to physically invert space to directly test conservation of parity. One can only prove that a specific model of nature is consistent or inconsistent with parity conservation. The original claims of parity violation ([e.g. Yang and Lee [1956], Wu [1957]) are simply based on dubious assumptions about the transformation properties of the Dirac equation.

The spatially inverted classical wave function must yield opposite sign for all of the velocity components. Complex conjugation can invert one component, and the other components can be inverted by rotation of π about the inverted component. If the polarization is a pseudo-vector then the spatially inverted classical wave function is:

$$P\psi(\mathbf{r}) = \frac{\dot{a}_0^{1/2}}{\sqrt{2}} \exp(-\mathrm{i}\,\boldsymbol{\sigma}\cdot\xi(-\mathbf{r})/2)\mathrm{i}\,\beta_2\left[\exp(\mathrm{i}\,\boldsymbol{\sigma}\cdot\xi(-\mathbf{r})/2)\psi(-\mathbf{r})\right]^* \tag{63}$$

The inverted wave function also satisfies the classical bispinor wave equation.

5. Waves in an Elastic Solid

Define an infinitesimal displacement field $\mathbf{a}(\mathbf{x},t)$ which represents displacement from equilibrium of material currently at position $\mathbf{x}$. The usual force equation for an ideal elastic solid with shear modulus μ and Lam□ constant λ is (see e.g. Morse and Feshbach [1953b]):

$$\mathbf{F} = \mu\nabla^2\mathbf{a} + [\mu+\lambda]\nabla[\nabla\cdot\mathbf{a}] = -\mu\nabla\times[\nabla\times\mathbf{a}] + [2\mu+\lambda]\nabla[\nabla\cdot\mathbf{a}] \tag{64}$$

It is customary to separate this equation into shear motion ($\nabla\cdot\mathbf{a}_\perp \equiv 0$) and compressional motion ($\nabla\times\mathbf{a}_\| \equiv 0$). However, a simple analysis shows that this separation cannot generally be maintained. If in some finite region the medium rotates uniformly with angular frequency **w,** then the centripetal acceleration must be attributable to compressional stress (the compressional force must balance the centrifugal force):

$$[2\mu+\lambda]\nabla[\nabla\cdot\mathbf{a}] = \rho_m \mathbf{w}\times\dot{\mathbf{a}} \tag{65}$$

where $\dot{\mathbf{a}} = \partial\mathbf{a}/\partial t$ and $\mathbf{w} \equiv \partial\boldsymbol{\theta}/\partial t = [\nabla\times\dot{\mathbf{a}}]/2$ is the vorticity. The acceleration represented by $\mathbf{w}\times\dot{\mathbf{a}}$ has no work associated with it because it is perpendicular to the velocity $\dot{\mathbf{a}}$. Hence there is no coupling to longitudinal waves in spite of non-vanishing compression. For non-rigid rotation we can introduce an unknown amount of compression to partially compensate for rotation using a free parameter $0 \le \gamma(\mathbf{r},t) \le 1$. We can still write an equation for $\mathbf{a}_\perp$ without explicit reference to $\mathbf{a}_\|$:

$$\rho_m\left[\frac{\partial^2\mathbf{a}_\perp}{\partial t^2} - \gamma(\mathbf{r},t)\mathbf{w}\times\dot{\mathbf{a}}_\perp\right] = -\mu\nabla\times[\nabla\times\mathbf{a}_\perp] = \mu\nabla^2\mathbf{a}_\perp \tag{66}$$

One limitation of the above force equation is that it was derived assuming infinitesimal displacements. Consider now a static medium with torsion $\partial\theta/\partial z = \text{constant}$. In cylindrical coordinates (r,ϕ,z) with rotation $\theta(z)$ about the z-axis:

$$\mathbf{a}_\perp = \hat{\mathbf{e}}_\phi a_\phi(z) = \hat{\mathbf{e}}_\phi r\theta(z)\ ;\ \ \nabla\times\mathbf{a}_\perp = \hat{\mathbf{e}}_z 2\theta(z) - \hat{\mathbf{e}}_r\,\partial\theta/\partial z\ \ ;\ \ \nabla\times(\nabla\times\mathbf{a}_\perp) = 0 \tag{67}$$

The infinitesimal field $\mathbf{a}_\perp(\mathbf{x},t)$ produces no net force on the medium. However, now consider the effect of finite rotations. A rotation by angle θ yields a radial displacement of:

$$a_r = r[\cos(\theta) - 1] \tag{68}$$

The first derivative is:

$$\frac{\partial a_r}{\partial z} = -r\sin(\theta)\frac{\partial\theta}{\partial z} \tag{69}$$

which vanishes at θ=0. However, the second derivative is finite:

$$\left.\frac{\partial^2 a_r}{\partial z^2}\right|_{\theta=0} = -r\left[\frac{\partial \theta}{\partial z}\right]^2 = -\frac{\partial \theta}{\partial z}\frac{\partial a_\phi}{\partial z} \tag{70}$$

Generalizing to arbitrary rotation direction yields the general nonlinear correction to $\nabla\times(\nabla\times\mathbf{a}_\perp)$:

$$\nabla\times(\nabla\times\mathbf{a})_{\text{nonlinear}} = -\nabla^2\mathbf{a}_{\text{nonlinear}} = \frac{\partial \boldsymbol{\theta}}{\partial x_i}\times\frac{\partial \mathbf{a}_\perp}{\partial x_i} \tag{71}$$

Multiplied by the shear modulus μ, this is the force which causes a long thin balloon to collapse radially when twisted. The tensor quantity $\tau_{ij} \equiv \partial\theta_j / \partial x_i$ is the torsion tensor, which for simplicity can also be written as $\boldsymbol{\tau}_i \equiv \partial\boldsymbol{\theta}/\partial x_i = \partial[\nabla\times\mathbf{a}_\perp/2]/\partial x_i$.

Adding this nonlinear correction to the force equation yields:

$$\rho_m\left[\frac{\partial^2\mathbf{a}_\perp}{\partial t^2} - \gamma(\mathbf{r},t)\mathbf{w}\times\dot{\mathbf{a}}_\perp\right] = \mu\left[\nabla^2\mathbf{a}_\perp - \boldsymbol{\tau}_i\times\frac{\partial\mathbf{a}_\perp}{\partial x_i}\right] \tag{72}$$

This is the wave equation for shear waves with speed $c=\sqrt{\mu/\rho_m}$ in a medium with rotation and torsion. This agrees with the vector wave equation (42) if we neglect the final term in that equation, set $\gamma(\mathbf{r},t)=1$, and equate rotations of polarization with rotations of the medium:

$$\frac{\partial\boldsymbol{\xi}}{\partial t} = \mathbf{w} \equiv \frac{1}{2}\frac{\partial}{\partial t}[\nabla\times\mathbf{a}] \; ; \; \frac{\partial\boldsymbol{\xi}}{\partial x_i} = \boldsymbol{\tau}_i \equiv \frac{1}{2}\frac{\partial}{\partial x_i}[\nabla\times\mathbf{a}] \tag{73}$$

These equalities are valid for rotational motion but are not always valid. For example, if the medium executes a rigid circular orbit without changing orientation then spatial derivatives are zero and **w**=0 but $\partial\boldsymbol{\xi}/\partial t \neq 0$ if the polarization represents displacement. However, such motion requires external forces. Conversely, planar shear waves have $\partial\boldsymbol{\xi}/\partial t = 0$ but $\mathbf{w}\neq 0$ if the polarization represents displacement.

Now consider angular momentum. The transverse velocity ($\dot{\mathbf{a}}_\perp$) is related to angular momentum density (**s**) by the relation:

$$\rho_m\dot{\mathbf{a}}_\perp = -\frac{1}{2}\nabla\times\mathbf{s} \tag{74}$$

Define a vector potential **Q**(**x**,*t*) such that $\dot{\mathbf{Q}}=\mathbf{s}$ (also $\nabla\times\mathbf{Q}=-2\rho_m\mathbf{a}_\perp$ and $\nabla\cdot\mathbf{Q}=0$). In the absence of rotation or torsion we could deduce the equation of evolution to be:

$$\frac{\partial^2\mathbf{Q}}{\partial t^2} = \frac{\mu}{\rho_m}\nabla^2\mathbf{Q} = c^2\nabla^2\mathbf{Q} \tag{75}$$

since the curl of this equation yields the appropriate equation for $\mathbf{a}_\perp$.

Correcting for rotation and torsion (using **ξ=θ** to represent the rotation angle) yields:

$$\frac{\partial^2\mathbf{Q}}{\partial t^2} - \mathbf{w}\times\dot{\mathbf{Q}} = c^2\left[\nabla^2\mathbf{Q} + \boldsymbol{\tau}_i\times\frac{\partial\mathbf{Q}}{\partial x_i}\right] \tag{76}$$

which has the same form as (42) if the final term in that equation is neglected.

The bispinor wave function associated with the wave equation for the vector potential **Q** has the property:

$$\psi^\dagger\boldsymbol{\sigma}\psi = \dot{\mathbf{Q}} = \mathbf{s} \tag{77}$$

The same operator (within a constant factor) represents the intrinsic (or spin) angular momentum density in quantum mechanics.

The vorticity and torsion can be derived directly from the wave function:

$$\mathbf{w} = \frac{\partial \boldsymbol{\theta}}{\partial t} = \frac{1}{4\rho_m} \nabla^2 \mathbf{s} = \frac{1}{4\rho_m} \nabla^2 \left[\psi^\dagger \boldsymbol{\sigma} \psi \right]$$

$$\boldsymbol{\tau}_i = \frac{\partial \boldsymbol{\theta}}{\partial x_i} = \frac{1}{4\rho_m} \nabla^2 \frac{\partial}{\partial x_i} \mathbf{Q} = -\frac{1}{4\rho_m c} \nabla^2 \left[\psi^\dagger \beta_i \boldsymbol{\sigma} \psi \right] \tag{78}$$

Assuming these account for rotation of polarization, the corresponding bispinor equation (37) is:

$$\frac{\partial}{\partial t}\psi + c\boldsymbol{\beta} \cdot \nabla \psi = -\frac{1}{2}\left[\mathbf{w} + c\beta_i \boldsymbol{\tau}_i \right] \cdot \mathrm{i}\,\boldsymbol{\sigma} \psi - \frac{1}{2}\left[\frac{\partial \boldsymbol{\zeta}}{\partial t} + c\boldsymbol{\beta} \cdot \nabla \boldsymbol{\zeta} \right] \cdot \tilde{\mathrm{i}}\, \boldsymbol{\beta} \psi \tag{79}$$

This is a classical example of a nonlinear bispinor equation. Others have proposed that elementary particles might represent soliton solutions of a nonlinear Dirac equation (see e.g. Ranada [1983] for a review). Hence this equation for shear waves in an ideal elastic solid represents a plausible basis for describing matter waves.

6. Wave Properties of Matter

Many physical properties of matter can be derived from the simple model of waves in an elastic solid. The Uncertainty Principle applies to all classical waves. Lorentz invariance is also a property of all waves. For example, the relativistic phenomenon of time dilation is simply explained by the fact that stationary standing waves execute circular (or linearly oscillating) orbits whereas moving waves execute spiral (or cycloidal or sinusoidal) orbits which have longer wave paths in each cycle [Close 2006]. Absolute motion with respect to the medium (the aether) is not detectable using matter waves because without prior knowledge of absolute motion it is unknown whether a signal is Doppler shifted at the source or at the receiver.

We have suggested in this paper that transverse waves in an elastic solid may produce compression which balances centrifugal forces. If the far field density profile decreases with distance from matter then the resulting spatial dependence of the wave speed has the effect of a gravitational field. General relativity (see e.g. Einstein [1956]) also predicts a spatially variable light speed, and other investigators (e.g. de Felice [1971]) have described gravity as a spatially varying index of refraction.

7. Conclusions

In this paper we have constructed first and second order equations describing vector waves. The bispinor wave function represents a parameterized solution of a vector wave equation in which the polarization and wave velocity evolve via convection and rotation. The wave function can be factored into constant matrix, a single amplitude, a one-dimensional Lorentz velocity boost, rotation of polarization (three factors), and rotation of wave velocity (three factors). The quantum mechanical factorization is similar but lacks a factor rotating velocity independently from polarization. Interpreting the classical bispinor equation as an equation for an electron, it is found that the mass term represents rotations of velocity and possibly polarization. Perturbations of the velocity angle yield electromagnetic potentials which automatically satisfy the Lorenz condition. The conventional parity operator is shown to represent a type of rotation rather than a reflection. This finding suggests that it should be possible to construct a theory of nature for which parity is conserved. The second order wave equation for displacements in an elastic solid includes terms which account for rotation and torsion of the medium. A classical nonlinear bispinor equation is proposed for matter-like waves based on the assumption that rotation of wave parameters is due entirely to rotation of the medium.

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