Three atomic quantities derived from an electrodynamics experiment in discharge condition (~ 40000 K)

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The results of an experiment of impulsive electrodynamics [N. Graneau, T. Phypps, and D. Roscoe, Eur. Phys. J. D **15**, 87 (2001)] are shown to be due to electrons and ions in runaways. By fitting the theoretical values with the experimental data, the values of atomic quantities, at present unknown, can be derived, thus opening a new field of research. The obtained atomic quantities are three, namely: i) the contribution to air ionization due to the current (mainly of runaways) and characterized by a parameter ρ; ii) the product $\zeta = n_{ei}n_{ie}$ (where n_{ei} is the number of ions extracted by one electron in runaway and n_{ie} the number of electrons extracted by one runaway ion colliding on the electrodes in electrical discharges with temperatures (for non runaways) of $\simeq 4 \times 10^4$ K); iii) the reconstruction time constant $\mathcal{T}$ of the high-energy tail of the distribution function, from which we can derive the concentration per unit time of electrons and ions which become runaways. The $\mathcal{T}$ value is useful for the theoretical explanation of the electronic noise with power spectral density inversely proportional to the frequency.

I. INTRODUCTION

Recently, Graneau *et al.* [1] have performed an interesting experiment of electrodynamics sketched in Fig. 1. A capacitor bank C was charged to a voltage φ_0 and connected, after a switch S_1, to a circuit whose sections are denoted by $1, 2, 3, 4, 5$. Section 1 ends upwards with a bottom gap while section 5 ends downwards with a top gap. A mobile section, called armature in Ref. [1], was initially placed with air gaps between it and the bottom electrode. By closing switch S_1 the potential difference φ_0 is applied to the two gaps in series after a very short time producing a discharge and an electric current through the circuit. The result is that the mobile armature was found to have moved upwards by an easily measurable amount when the length of the bottom gap was smaller than that of the upper gap.

Referring to a previous paper [2] for the criticism of Graneau *et al.* [1] interpretation relevant to their own experiment, we show in the present paper that the longitudinal forces that raise the mobile armature are caused by the collisions of electrons and ions in runaway, striking the bases of the armature. Since the majority of these particles are extracted with initially high transversal velocities by other electrons and ions in runaway, a large fraction of them can collide on one basis of the armature if the corresponding gap is very small. If the gap is large, many runaway electrons and ions escape laterally outside the gap, and the impulses, and consequent pressure, are smaller than those in the smaller gap. The theoretical predictions depend on 3 parameters, namely: i) the contribution to air ionization due to the current (mainly of runaways) and characterized by a parameter ρ; ii) the product $\zeta = n_{ei}n_{ie}$ (where n_{ei} is the number of ions extracted by one electron in runaway and n_{ie} the number of electrons extracted by one runaway ion colliding on the electrodes in electrical discharges with temperatures (for non runaways) of $\simeq 4\times10^4$ K); iii) the reconstruction time constant $\mathcal{T}$ of the high-energy tail of the distribution function, from which we can derive the concentration per unit time of electrons and ions which become runaways. It is therefore sufficient a data fitting with 3 experimental results to obtain the wanted parameters. Actually, we have 15 data and the redundance improve the accuracy of the derived values. It is difficult to measure by other means those physical quantities which are relevant to very high temperatures in electrical discharges and, at present, their values are unknown in atomic physics. The point is that Graneau *et al.* [1] have performed an experiment completely unaware of the capacity and range of it. We encourage Graneau, Phipps Jr., and Roscoe to improve their experiment that can open a new stream of atomic physics. Some manipulations and improvements of the experimental data have been done by us in this

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paper, which is organized as follows.

In Sec. II we accurately examine the electrical circuit of Ref. [1] and, resting on the most reliable quantities measured in Ref. [1], we correct the values of some other quantities. We will see that these corrections partially explain the strong fluctuations in the experimental results reported in Ref. [1].

In Sec. III, we evaluate the time required to reach a rarefaction of the air at which the air can no longer be considered as a macroscopic continuum. This time turns out to be roughly 1/60 of the period of the underdamped oscillating current. During this time interval the pressures in the two gaps are practically the same and there is no net effect for the impulsive force on the mobile armature. In this section we use the fraction f_T of ionization only due to the thermal motion at the temperature T, i.e., we set $\rho = 0$ where ρ is the first unknown parameter.

In Sec. IV we consider the conditions of the air after the time at which its particulate constitution is predominant and the mean free paths of many ions and electrons become larger than the lengths of the gaps. These ions and electrons are in runaway and, colliding on the electrodes, extract other ions and electrons with large transversal velocities $v_\perp$, which become new runaways. Because of their large $v_\perp$, a much larger fraction of them does not hit one basis of the mobile armature in the longer gap than in the shorter gap. The consequent different pressures in the two gaps is the cause of the net impulse communicated to the mobile section (or armature) of Graneau *et al.*'s circuit.

In Sec. V we apply the theoretical results found in Sec. IV to find the net impulse communicated to the mobile armature by the runaway ions and electrons. The data fitting with the experimental values allows the derivation of the two unknown parameters. Actually, the procedure is iterative, i.e., we first find a first order value ρ_1 for ρ. Then we recalculate all the quantities already evaluated in Sec. III with zero-order approximation. With the new values we obtain a second order value ρ_2 that is shown to be the final value (the iterative procedure is rapidly convergent).

We conclude in Sec. VI.

II. CORRECTED PARAMETERS OF THE ELECTRICAL CIRCUIT

The circuit used in Ref. [1] and shown in Fig. 1 and 2 can be schematized by a large capacitor C, initially charged with a voltage φ_0, followed by a switch S_1, an inductor L in series with a resistor R_1, in turn in series with the two gaps sinthetized by a small capacitor C_g having in parallel a switch S_2 followed by a resistor $R_2(I)$, function of the current I in the circuit. After closing switch S_1 the voltage across the two gaps rises to the value φ_0 after a very short time t_1. Then a discharge occurs across C_g which is equivalent to the closure of switch S_2. At this point the presence of C_g is negligible

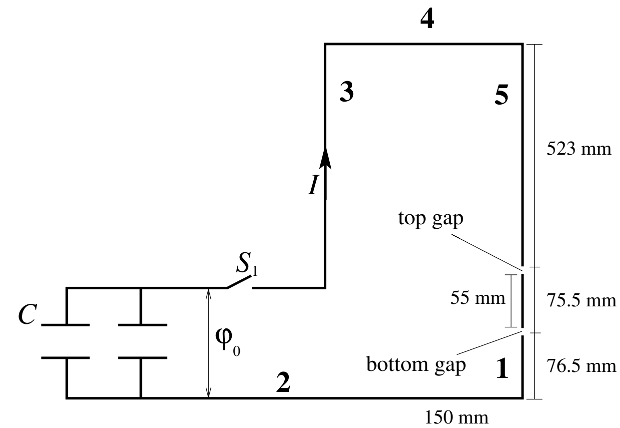

FIG. 1: Sketch of the electrical circuit used by N. Graneau *et al.* [1]. The capacitors C have been charged so as to have a potential difference φ_0. Once closed the switch S_1 the potential difference φ_0 is applied to the two gaps in series after a very short time t_1. Then, after another short time $t_0 - t_1$ a discharge across the two gaps make a current I flows in the circuit. The result is that the mobile section of length 55 mm receives an impulsive force toward high if the length of the bottom gap is smaller than that of the top gap.

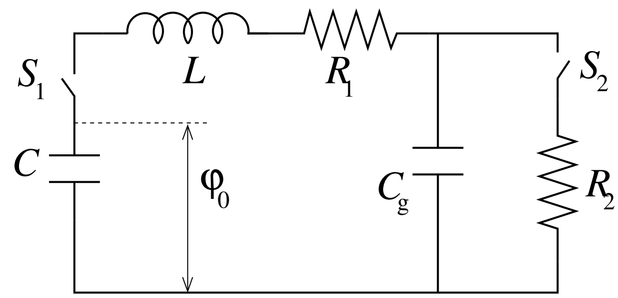

FIG. 2: Electrical scheme of the circuit used in Ref. [1]. A large capacitance C is initially charged with a voltage φ_0. When the switch S_1 is closed, the voltage across the small capacitance C_g reaches the value φ_0 in a very short time t_1. A discharge across the gap represented by C_g is equivalent to close the second switch S_2 having a resistance R in series. After the closure of S_2 we can neglect C_g and the circuit can be reduced to a series of a capacitor C, an inductor L, and a resistor $R = R_1 + R_2$.

and the practical circuit is equivalent to an inductance L in series with a resistance $R = R_1 + R_2(I)$, and a capacitance C initially charged with a voltage φ_0. The circuit is underdamped and the current, after the ignition of the discharge and supposing $R_2(I)$ independent of I, is expressed by

$$I = \frac{\varphi_0 \sqrt{C/L}}{\sqrt{1 - R^2C/(4L)}} \exp\left(-\frac{Rt}{2L}\right) \times \sin\left(\frac{t}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}}\right) . \quad (1)$$

The minimum capacitance used in Ref. [1] is $C_m = 3.34\ \mu$F, the measured angular frequency ω is the maximum one $\omega_M = 3.4 \times 10^5\ \mathrm{s}^{-1}$ so that the inductance must be, if $R^2C/(4L) \ll 1$,

$$L = (C\omega^2)^{-1} = 2.59\ \mu\mathrm{H}\ , \tag{2}$$

instead of their [1] estimated value $L = 2.8\mu$H. Their measured time constant τ of the decay (5th column of table I of Ref. [1]) is $\tau = 54.3\ \mu$s. Consequently, we derive from Eq. (1), since $\tau = 2L/R$,

$$R = \frac{2L}{\tau} = 9.54 \times 10^{-2}\ \Omega\ , \tag{3}$$

so that

$$R^2C/(4L) = 2.93 \times 10^{-3}\ . \tag{4}$$

Since this value is much less than unity, the correction to Eq. (2) is in the fourth significant figure. With this approximation, Eq. (1) reduces to

$$I \simeq \varphi_0 \sqrt{\frac{C}{L}} \exp\left(-\frac{Rt}{2L}\right) \sin(\omega t)\ . \tag{5}$$

The power dissipated in the gaps if $R_1 \ll R_2(I) \simeq R$ is therefore expressed by

$$P = RI^2 = P_0 \exp\left(-\frac{2t}{\tau}\right) \sin^2(\omega t)\ , \tag{6}$$

where

$$P_0 = R\varphi_0^2 C/L\ , \tag{7}$$

the decay time constant τ being defined in Eq. (3).

As derivable from Eq. (1), the amplitude of I for $t \to 0$ is $I_0 \simeq \varphi_0\sqrt{C/L}$, which has been measured and reported in the third column of table I of Ref. [1]. It is $I_0 = 42.9$ kA, whence

$$\varphi_0 = I_0\sqrt{L/C} = 37.7\ \mathrm{kV}\ , \tag{8}$$

instead of the value $\varphi_0 = 33$ kV reported in the first column of table I of Graneau *et al.* [1]. We confirm the two corrected values (of L and φ_0) by the agreement between the values of the dissipated power P calculated in two different ways. In the first way we express P by the Joule dissipated power averaged over half a period T

$$\begin{aligned} \langle P\rangle_{T/2} &= \langle RI^2\rangle_{T/2} \simeq \frac{1}{2}P_0 \exp\left(-\frac{2t}{\tau}\right) \\ &= 8.74 \times 10^7 \exp\left(-\frac{2t}{\tau}\right)\ \mathrm{W}\ . \end{aligned} \tag{9}$$

In the second way we start from the energy $\mathcal{E}$ stored in C

$$\mathcal{E} = \frac{1}{2}C\varphi_0^2 = 2.37 \times 10^3\ \mathrm{J}\ . \tag{10}$$

Since

$$\mathcal{E} = \int_0^\infty P\,dt = \int_0^\infty \frac{1}{2}P_0 \exp\left(-\frac{2t}{\tau}\right) dt = P_0\tau/4\ , \tag{11}$$

we obtain

$$\frac{1}{2}P_0 = 2\mathcal{E}/\tau = \frac{2 \times 2.37 \times 10^3}{5.43 \times 10^{-5}} = 8.73 \times 10^7\ \mathrm{W}\ , \tag{12}$$

in excellent agreement with Eq. (9). If we had kept $\varphi_0 = 33$ kV (as written by N. Graneau *et al.* [1]) there would have been a disagreement by a factor 1.3.

Analogous corrections have been carried out for all the other values of capacities and gap lengths and are reported in Table I. Notice that P_0 can be calculated either by Eq. (9) or by Eq. (12). The values calculated using Graneau data with $\varphi_0 = 33$ kV lead to a disagreement up to 30%. On the contrary, the pairs of values calculated using our corrected values differs from each other of less than 0.5%.

The preceding treatment is valid with good approximation after the ignition of the discharge. Before the ignition, the electrical resistance between the gaps is practically infinite and we have a small capacitance $C_g \simeq 10$ pF (due to the gaps and the conductors of the circuit) in series to the large capacitance $C \simeq 6\ \mu$F of the capacitor bank. Consequently, the voltage across the two gaps reaches the initial value φ_0 of the capacitor bank in a time

$$t_1 = \frac{\pi}{2}\sqrt{LC_g} = \frac{\pi}{2}\sqrt{LC}\sqrt{C_g/C} = 1.3 \times 10^{-3}\frac{\pi}{2\omega}\ , \tag{13}$$

which is roughly one thousand of a quarter of the damped current period.

The electric field $\mathbf{E}$ is practically equal in the two gaps and accelerates the free electrons extracted from the cathode because of tunnel effect through the decreased potential barrier. Once the discharge is ignited, the current in the two gaps in series is the same. Moreover, before the formation of the runaways, the cross-sections of the ionized air gaps are the same and such are their resistivities. Consequently, the accelerated free electrons ionize other atoms and an avalanche is brought about. The space charge of any avalanche is due to a spherical ball of electrons and to a conical envelope containing the positive ions. The electric field $\mathbf{E}_c$ due to the space charge of any avalanche on its symmetry axis is parallel to the external field $\mathbf{E}$ outside the avalanche and antiparallel inside. As the avalanche grows, the total field $\mathbf{E}+\mathbf{E}_{ci}$ inside the avalanche decreases until it practically vanishes. Just outside the avalanche (and on its symmetry axis), $\mathbf{E}+\mathbf{E}_c \simeq 2\mathbf{E}$ and other free electrons (produced by photoionization of impurities) in the two regions around the two ends of the avalanche, are strongly accelerated and bring about other avalanches. When all the avalanches join together and reach the electrodes, a high density current ("streamer" process) begins to flow in the initial ionized channel behaving as a very thin wire connecting

l_1 (mm)	τ (μs)	$R10^2$ (Ω)	I_0 (kA)	φ_0 (kV)	$\mathcal{E}$ (kJ)	$\frac{1}{2}P_0\,10^{-7}$ (W)
$C = 3.34$ (μF) ; $\omega = 3.4 \times 10^5$(rad/s) ; $L = 1/(C\omega^2) = 2.59$ (μH)						
1.0	54.3	9.54	+42.9	+37.8	2.39	8.80
2.0	54.3	9.54	+42.9	+37.8	2.39	8.80
$C = 5.01$ (μF) ; $\omega = 2.8 \times 10^5$(rad/s) ; $L = 2.55$ (μH)						
1.0	66.2	7.7	+54.6	+38.9	3.80	11.5
1.0	69.6	7.3	-51.5	-36.7	3.38	9.8
2.0	65.3	7.81	+53.1	+37.9	3.60	11.0
2.0	64.9	7.86	-51.9	-37.0	3.43	10.6
$C = 6.68$ (μF) ; $\omega = 2.5 \times 10^5$(rad/s) ; $L = 2.39$ (μH)						
1.0	71.9	6.65	+62.3	+37.3	4.65	12.9
2.0	69.0	6.93	+60.9	+36.4	4.43	12.8
2.0	75.1	6.40	-60.3	-36.1	4.35	11.6
4.0	67.1	7.12	+63.5	+38.0	+4.82	14.4
5.0	63.7	7.50	+63.3	+37.9	4.80	15.1
$C = 8.35$ (μF) ; $\omega = 2.2 \times 10^5$(rad/s) ; $L = 2.47$ (μH)						
3.0	78.9	6.30	-68.7	-37.4	5.84	14.8
4.0	68.0	7.26	+71.2	+38.7	6.25	18.4
$C = 10.02$ (μF) ; $\omega = 2.0 \times 10^5$(rad/s) ; $L = 2.48$ (μH)						
4.0	81.3	6.10	+75.0	+37.2	6.97	17.1
8.0	74.6	6.65	+76.5	+38.0	7.27	19.5
10.2	74.6	6.65	+76.5	+38.0	7.27	19.5

TABLE I: Corrections of the values reported in Table I of Ref. [1]. In this our table we have considered: $R = \frac{2L}{\tau}$; $\varphi_0 = I_0\sqrt{\frac{L}{C}}$; $\mathcal{E} = \frac{C}{2}\varphi_0^2$; $\frac{1}{2}P_0 = \frac{2\mathcal{E}}{\tau}$.

the electrodes. It is this very concentrated discharge that causes the ablations noted in Ref. [1] and also a shock wave with gas and discharge expansion.

The time t_0 taken to produce the ionized channel, and therefore to produce the initial discharge, is the sum of t_1 given by Eq. (13) plus the time taken by the streamer to cross the total length of the two gaps ($l = l_1 + l_2 = 20.5$ mm), i.e.,

$$t_0 = t_1 + l/v_{\rm streamer} \; . \tag{14}$$

Now the propagation of the streamer is not simply limited by the drift velocity $w(E)$ where $\mathbf{E}$ is the electric field in the gaps. First, while the avalanches are partially formed, the electric field outside them increases and, on an average, it is $2E$. Second, the propagation is mainly due to photoionization, and this implies another factor 3. Consequently, $v_{\rm streamer} \simeq 6w(E)$. The drift velocity w is given by

$$w = \frac{eE}{m}\left\langle \frac{1}{\nu_i} - \frac{v}{3\nu_i^2}\frac{d\nu_i}{dv}\right\rangle \simeq 2\frac{eE}{m}\left\langle\frac{1}{\nu_i}\right\rangle \; , \tag{15}$$

e and m being the electron charge and mass, respectively, $\nu_i = \nu_i(v)$ the electron collision frequency which is dominated by the ion interactions, and E the electric field that can be taken as $E \simeq \varphi_0/l$ since, as φ decreases until connecting to RI, the distances between the avalanches decrease. After numerical calculations, Eq. (15) yields

$$w = \frac{17.9\varphi_0\sqrt{2\pi}\varepsilon_0^2(kT_{\rm streamer})^{3/2}}{lN_0e^3m^{1/2}} \; , \tag{16}$$

where $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1} the vacuum permettivity, $k = 1.38 \times 10^{-23}$ JK^{-1} the Boltzmann constant, and $T_{\rm streamer} \simeq 4 \times 10^4$ K the absolute temperature of the streamer avalanches with our E values. The uncertainty is of little importance because an 100% error in w, hence in t_0, implies a 1% error in the predictions of the height h reached by the mobile armature. The value of the initial concentration is $N_0 = 2N_a$ where $N_a = 2.7 \times 10^{25}$ m^{-3} is the air concentration at sea level and at $T_a \simeq 300$ K, the factor 2 being due to the immediate dissociation of the air molecules as soon as the discharge begins. The other quantities in Eq. (16) takes the values: $l = 2.05 \times 10^{-2}$ m, $e = 1.6 \times 10^{-19}$ C, and $m = 9.11 \times 10^{-31}$ kg. With those value for the temperature we obtain $w \simeq 1.24 \times 10^4$ m/s, whence from Eqs. (13) and (14) with $v_{\rm streamer} \simeq 6w$,

$$t_0 = (8.2 \times 10^{-9} + 2.7 \times 10^{-7})\text{ s} = 2.8 \times 10^{-7}\text{ s} \; , \tag{17}$$

practically independent of the C values.

III. EVALUATION OF THE TIME INTERVAL DURING WHICH THE AIR IN THE GAPS BEHAVES AS A MACROSCOPIC CONTINUUM

Denoting $p_1(t)$ and $p_2(t)$ the pressures as functions of time t in the two gaps, respectively, the net impulse on the mobile rod (or armature of mass M_A) of the circuit is given by

$$M_A v = \pi r_g^2 \int_0^\infty dt[p_1(t) - p_2(t)] \ , \tag{18}$$

where $r_g = 2.38$ mm is the radius of the conductors (including the mobile rod) delimiting the two gaps. To obtain $p(t)$ we must also calculate the air molecule concentration $N(t)$, the air temperature $T(t)$, the velocity $V(t)$ of the air expansion, the fraction $f(t)$ of the ionized atom, the expansion velocity of the discharge channel. Fortunately, the latter quantity can be taken as equal to the velocity V of the air expansion on the front of the shock wave. Indeed, as soon as the air density decreases inside the expanding shock wave, the electron mean free paths increase and the ionization is favoured. We have therefore 5 variables, namely p, N, T, V, f and we need five equations. They are: 1) the equation for a perfect gas $p = NkT$, 2) the Boltzmann weight factor f_T plus the contribution to ionization due to the current I, 3) the Euler equation of motion for a perfect fluid, 4) the continuity equation, 5) the energy balance. Equations 1) and 2) are algebraic while 3), 4), and 5) are nonlinear differential equations.

A. Air as a macroscopic fluid

Let us first treat the air as a macroscopic fluid. As soon as the electrical discharge is triggered in one gap, the voltage across it decreases while it increases in the other gap, until an equal current flows in the two gaps connected in series. If in these conditions the air temperature in one gap is smaller than in the other, the ionization decreases, the electrical resistance R increases so that the power RI^2 injected in this gap increases until the two temperatures in the two gaps become equal. With the same T the two pressures p_1 and p_2 are the same so that Eq. (18) gives no net effect. However, when the air density $N(t)$ has dropped around a value that in the next sections we have estimated to be $N_a/86$, while the temperature T still remains very high, we must consider the particulate aspect of the air (ions and electrons) since the more energetic electrons and ions acquire mean free paths of the same order as the gap radius r_g. At this level, studied in the next Sec. IV, it is $p_1(t) > p_2(t)$ if $l_1 < l_2$ where l_1 and l_2 are the lengths of gaps 1 and 2, respectively. The aim of this section is only to calculate the time interval t^* to reach the particulate aspect. Since in this first phase p, N, and T are equal in the two gaps, for simplicity we calculate t^* for a single gap of length $l = l_1 + l_2$, so as to use the total power P injected in them.

A gas at high temperature behaves as a perfect gas, so that we can use

$$p = NkT \ . \tag{19}$$

To the aim of calculating both the runaway current in the gaps and the energy absorbed in heating the air during the discharge, we find expressions for the average number of electrons extracted from one atom and the average energy per atom necessary to have multiple ionizations. The fraction f_{Ts} of ionized atoms in the sth level of ionization and due only to the temperature T is given by the Boltzmann weight factor. Consequently, the total number f_T of free electrons per atom (the molecules are practically all dissociated at the discharge temperature) and only due to the thermal ionization, is given by the sum of the ionization factors

$$f_T = \sum_{s=1}^{7} \exp(-\epsilon_{is}/kT) \ . \tag{20}$$

where ϵ_{is} is the ionization energy of the sth level per one neutral atom. We have extended the sum to complete ionization of nitrogen and to all but one levels for oxygen (we favour the nitrogen atomic number since nitrogen percentage in air is $\simeq 80\%$).

The energy required to produce the ionization per atom at temperature T is expressed by

$$\mathcal{E}_i = \sum_{s=1}^{7} \epsilon_{is} \exp(-\epsilon_{is}/kT) \ . \tag{21}$$

Even at the discharge temperatures the second, third, etc. ionizations per atom are negligible. Consequently, in the following we only take $s = 1$ in both Eqs. (20) and (21). The ionization due to the current I makes the number f of free electrons increase until complete ionization is reached asymptotically (for $I \to \infty$). For nitrogen, complete ionization implies $f = 7$, and the very complicated process can be summarized as

$$f = f_T + (7 - f_T)\left\{1 - \exp\left[-\frac{1}{2}(\rho I_0)^2 \exp\left(-t/\tau\right)\right]\right\} \ , \tag{22}$$

where ρ is an arbitrary parameter to be determined by data fitting of the theoretical with the experimental results. We use therefore an iterative procedure, first putting $f = f_T$ in this section, in order to calculate the time t^* necessary to reach the particulate conditions (where diffusion dominate and we can have runaways electrons). Then, in Sec. IV, we use Eq. (22) to express the height h reached by the mobile armature. The data fitting (performed in Sec. V) with the experimental data gives a first order value ρ_1 for ρ. With ρ_1 we recalculate t^* and the other quantities of Sec. III, which change in a modest way. The new data fitting with the experimental data gives a second order value ρ_2 with a small difference with respect ρ_1. The use of ρ_2 in the expressions of Sec. III leads to a new t^* and other quantities with no appreciable difference from those derived with ρ_1. The procedure converges very quickly and we take ρ_2 as the final value.

The classical equation of motion for perfect fluids, called Euler's equation, is

$$\mathbf{f}_m - \frac{1}{Nm_a}\boldsymbol{\nabla} p = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} \ , \qquad (23)$$

$\mathbf{f}_m$ being the force per unit mass, which can be approximated, for gravity and small variations of altitude, with $-g\hat{\mathbf{e}}_z$ where g is the gravity acceleration and $\hat{\mathbf{e}}_z$ the unit vector of the vertical z axis. The electrical discharge is almost vertical and there is axial symmetry along the z axis so that $\mathbf{V} = V\hat{\mathbf{e}}_r$ and $\boldsymbol{\nabla} = \hat{\mathbf{e}}_z(\partial/\partial z) + \hat{\mathbf{e}}_r(\partial/\partial r)$. Consequently, projecting Eq. (23) on $\hat{\mathbf{e}}_r$, we obtain

$$-\frac{1}{Nm_a}\frac{\partial p}{\partial r} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} \ . \qquad (24)$$

The flux of molecules through a cylinder of fixed radius $r < r_g$ is opposite to the derivative of the molecule number contained in it

$$NV2\pi rl = -\frac{\partial}{\partial t}(N\pi r^2 l) \ , \qquad (25)$$

whence we derive the desired continuity equation for any fixed r value

$$V = -\frac{r}{2N}\frac{\partial N}{\partial t} \ . \qquad (26)$$

This result is valid from the Eulerian point of view (fixed r). If we use the Lagrangian point of view, following the expanding front, the total number of particles contained in it is conserved, i.e.,

$$0 = d(N\,\pi\, r_F^2 l)/dt \ , \qquad (27)$$

or

$$N = N_0\,(r_0/r_F)^2 = 2N_a(r_0/r_F)^2 \ , \qquad (28)$$

where r_0 is the initial radius of the columnar discharge, r_F the radius of the expanding front, and $N_0 = 2N_a$ since practically all the diatomic molecules (of O_2 and N_2) are dissociated inside the discharge, so that as soon as the discharge is ignited, the initial monoatomic molecule concentration is twice the air molecule concentration N_a at room temperature. The advantage of the Lagrangian point of view is to reduce the differential Eq. (26) to the algebraic Eq. (28).

1. *Energy balance*

As said after Eq. (22), in this section we consider the ionization as due only to the temperature of the air (i.e., neglecting the contribution of the electric current), which implies only some percents of ionization in the first level. We take for the ϵ_{i1} of the first level an average, weighted value of those of nitrogen and oxygen, which turns out to be $\epsilon_{i1} = 16.06$ eV $= 2.25\times10^{-18}$ J. Using Eq. (21) with only $s = 1$ [see what said just after Eq. (21)] and neglecting the room temperature (compared to the very high discharge temperature), the energy balance applied to a cylinder of radius $r < r_g$ is

$$P = A\sigma_S T^4(2\pi r\,l + \pi r^2) + \frac{d}{dt}\left[N\pi r^2 l\,(c_m\,m_a\,T + \epsilon_{i1}\exp(-\epsilon_{i1}/kT))\right] \qquad (29)$$

where $l = l_1 + l_2 = 20.5$ mm is the total length of the two gaps, P the injected power. The first term inside the round brackets in the r.h.s. of Eq. (29) represents the radiated power through the lateral surface of the discharge column, $\sigma_S = 5.67\times10^{-8}$ W/(m^{2}K^4) being the Stefan-Boltzmann constant, A the emission coefficient of the air. The latter does not depend on the air concentration N and the absolute temperature T since the attenuation distance of e.m. waves in the discharge is very small. The second term inside the round brackets in the r.h.s. of Eq. (29) represents the radiated power through the two bases, having assumed a reflection coefficient 0.5. The first term inside the square brackets represents the time derivative of the thermal energy stored inside the discharge, $c_m \simeq 357$ J (kg K)$^{-1}$ being the specific heat per unit mass of dissociated air at constant volume [3], and $m_a = 14.4$ (in atomic unit 1.66×10^{-27} kg) the average atomic mass of dissociated air. The second term inside the square bracket denotes the time derivative of the ionization (plus excitation and molecule dissociation) energy of the air neutral atoms.

At the beginning, P is not given by Eq. (6) since, when the current is practically zero, all the voltage is applied to the gap. However, as soon as the discharge begins, the current is dominated by the inductance and we can take Eq. (1) as reliable although the resistance R is variable and very high at the beginning of the discharge. But the voltage across the two gaps, expressed by $\varphi = R(I)\,I$, is far from being proportional to the current. The voltage, as said just before Eq. (13), rises to the maximum value φ_0 after the very short time t_1 and then decreases as $R(I)$ decreases with the increase of the current I. A good approximation consists in taking an exponential relaxation of φ until reaching the value RI for a time t less than an eight of a period, i.e., for $t < \pi/(4\omega)$. Then, after such time, we take an effective value gradually decreasing with the time constant 2τ starting from the value $RI_{\rm eff} = RI_0/\sqrt{2}$ which, taken from Eq. (5), is $R\varphi_0(C/2L)^{1/2}$. We therefore obtain

$$\varphi = \left[\varphi_0\exp\left(-\frac{2.1t}{t_0}\right) + RI_0\sin\omega t\right]\Theta\left(\frac{\pi}{4} - \omega t\right) + \frac{RI_0}{\sqrt{2}}\exp\left(-\frac{t}{2\tau}\right)\Theta\left(\omega t - \frac{\pi}{4}\right) \ , \qquad (30)$$

where $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$, while t_0 is given by Eq. (14).

We make the same separation for the current, i.e.,

$$I = I_0 \sin(\omega t)\Theta\left(\frac{\pi}{4} - \omega t\right) + \frac{I_0}{\sqrt{2}} \exp\left(-\frac{t}{2\tau}\right)\Theta\left(\omega t - \frac{\pi}{4}\right) , \tag{31}$$

with $I_0 = \varphi_0\sqrt{C/L}$.

Using Eqs. (30) and (31), the expression of the power injected into the two gaps is therefore

$$P(t) = \varphi I = \varphi_0^2\sqrt{\frac{C}{L}}\sin\omega t\left[\exp\left(-\frac{2.1t}{t_0}\right) + \frac{RI_0}{\varphi_0}\sin\omega t\right] \times\Theta\left(\frac{\pi}{4} - \omega t\right) + \frac{1}{2}R\varphi_0^2\frac{C}{L}\exp\left(-\frac{t}{\tau}\right)\Theta\left(\omega t - \frac{\pi}{4}\right) , \tag{32}$$

where, as in Eq. (30), we have approximated $\exp(-2.1t/\tau) \simeq 1$ for $\omega t < \pi/4$ since $\tau \simeq 24\pi/(4\omega)$. We see in the following that Eq. (32) is useful for the solution of Eq. (29). Actually, during this phase of the discharge there is a very rapid variation of T and of the number f of extracted electrons, while the number of molecules $\mathcal{N} = N\pi r^2 l$ does not change as expressed by Eq. (27). Consequently, the third and fourth terms at the r.h.s. of Eq. (29) are dominant. We can therefore use an iterative method, neglecting in first approximation the first two terms at the r.h.s. of Eq. (29) in the first stage of the discharge, i.e., from $t = 0$ to $t = t_0$. Integrating Eq. (29) with the use of Eq. (32) we obtain

$$\begin{aligned}\int_0^{t_0} P\,dt &= \frac{\varphi_0^2 t_0}{4.41 + \omega^2 t_0^2}\sqrt{\frac{C}{L}}\,[\omega t_0 - 2.1\exp(-2.1)\sin(\omega t_0) \\ &\quad - \exp(-2.1)\omega t_0\cos(\omega t_0)] \\ &\quad + \frac{\varphi_0^2 RC}{2L\omega}\,[\omega t_0 - \sin(\omega t_0)\cos(\omega t_0)] \\ &= N\pi r_0^2 l\,[c_m m_a T + \epsilon_{i1}\exp(-\epsilon_{i1}/kT)] ,\end{aligned} \tag{33}$$

where t_0 is given by Eq. (17) and $N = 2N_a = 5.4 \times 10^{25}$ m^{-3}.

In the second phase of the discharge, characterized by the expansion, Eq. (32) leads to a slow decrease of the temperature and the number $\mathcal{N}$ of air molecules still remains constant because of Eq. (27) until the discharge front r_F remains smaller than the gap radius r_g. We can therefore neglect the two terms under differentiation of Eq. (29), take $r = r_F$ so that Eq. (29) reduces to

$$P = A\sigma_s T^4\left(2\pi r_F l + \pi r_F^2\right) . \tag{34}$$

At the beginning of the expansion it is $r_F = r_0$ and we derive from Eq. (34)

$$r_0 = l\left[\left(1 + \frac{P(t_0)}{\pi A l^2 \sigma_s T^4}\right)^{1/2} - 1\right] . \tag{35}$$

The system of Eqs. (33) and (35) gives r_0 and $T(r_0)$ in first approximation. Substituting Eq. (35) into Eq. (33) with $N(r_0) = 2N_a$ we obtain a transcendent equation in T only. Introducing the numerical values $l = 2.05 \times 10^{-2}$ m, $r_g = 2.38 \times 10^{-3}$ m, $N_a = 2.7 \times 10^{25}$ m^{-3}, $m = 9.11 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C, $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}, $m_a = 14.4 \times 1.66 \times 10^{-27}$ kg$= 2.39 \times 10^{-26}$ kg, $c_m = 357$ J kg^{-1} K^{-1}, $A \simeq 0.4$ (for ionized air, $k = 1.38 \times 10^{-23}$ JK^{-1}, $\sigma_s = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, $\epsilon_{i1} = 2.25 \times 10^{-18}$ J (equivalent first ionization energy for air), $P(t_0)$ given by the first term of Eq. (32) with the data inside it taken from Table I for $C = 6.68$ μF, we obtain numerically, for the case $l_1 = 1$ mm,

$$T(r_0) = T_0 \simeq 4.24 \times 10^4 \text{ K} . \tag{36}$$

With this value, we derive from Eq. (35)

$$r_0 \simeq 2.14 \times 10^{-3} \text{ m} = 2.14 \text{ mm} . \tag{37}$$

At this point we calculate the energy radiated in the first stage, supposing $T = Qt$ and keeping the first order values for t_0 and r_0. We obtain

$$\int_0^{t_0} A\sigma_s(2\pi r_0 l + \pi r_0^2)(Qt)^4 dt = A\sigma_s(2\pi r_0 l + \pi r_0^2)\frac{T_0^4}{5}t_0 , \tag{38}$$

to be subtracted from the l.h.s. of Eq. (33). With the new net injected energy, and keeping the same t_0 values for minimum, intermediate, and maximum values of the capacitances used by Graneau *et al.* [1], we obtain

$$C = 3.34\ \mu F : \quad T_0 \simeq 4.44 \times 10^4 \text{ K} ; \quad r_0 \simeq 1.88 \text{ mm} . \tag{39}$$

$$C = 6.68\ \mu F : \quad T_0 \simeq 4.37 \times 10^4 \text{ K} ; \quad r_0 \simeq 1.93 \text{ mm} . \tag{40}$$

$$C = 10.02\ \mu F : \quad T_0 \simeq 4.37 \times 10^4 \text{ K} ; \quad r_0 \simeq 1.95 \text{ mm} . \tag{41}$$

We see that the differences between the extreme cases (maximum and minimum used capacitances) are very small.

From Eq. (34) written for r_F, T and r_0, T_0, respectively, we obtain

$$T = T_0 \left(\frac{2r_0 l + r_0^2}{2r_F l + r_F^2} \right)^{1/4} \left(\frac{P}{P(t_0)} \right)^{1/4} . \qquad (42)$$

2. *Expansion velocity*

We have now T as a function of the front radius r_F and of t through $P(t)$. The connection between r_F and $t(r_F)$ can be found if we obtain the velocity V of expansion as a function of r_F and $T[r_F, t(r_F)]$. Since there are no priviledged points inside the discharge, N, T, and p are uniform from $r = 0$ to $r_M = r_F - \Delta$. Then, they have rapid variations in the small interval Δ, shown in Fig. 3a)

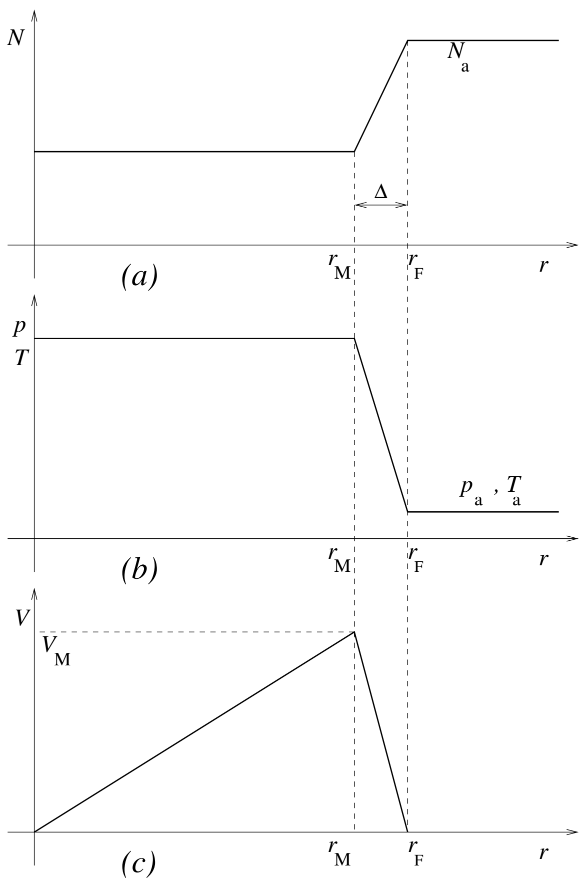

FIG. 3: Air concentration N [m^{-3}], pressures p [Nm^{-2}], temperature T [K], velocity V [ms^{-1}] vs the radius r [m] of the cylindrical expansion of the discharge at a fixed time t. The value r_F corresponds to the front of the shock wave and r_M is the preceding value at which the very rapid variations of the N, p, T, V begin. The distance $\Delta = r_F - r_M$ is very small compared to r_F and, for simplicity, we approximate the rapid variations of N, p, T, V inside Δ by straight segments. Since there are no priviledged points inside r_M, it follows that N, p, T are independent of r for $r \leq r_M$, while V increases linearly with r, the expansion being similar to Hubble's.

and b). The only dependence of V on r_F remains to be found. This can be achieved by solving Euler Eq. (24). The expansion is similar to the Hubble expansion of the universe (although with cylindrical symmetry instead of spherical) for $r < r_M$. Then we take V to go linearly to zero in the small interval $r_M \leq r \leq r_F$, as shown in Fig. 3c). We have therefore for $r \leq r_M$,

$$N(r) = N(r_M) = N_M \; ; \qquad p(r) = p(r_M) = p_M \; ; \qquad V = rH(t) \; , \qquad (43)$$

while, for $r_M \leq r \leq r_F$, it is

$$N = \frac{1}{2}(N_a + N_M) + \frac{1}{2}(N_a - N_M) \left(\frac{r - r_M}{\Delta} - \frac{t - t_F}{\Delta} V_M \right) , \qquad (44)$$

$$V = V_M \left(1 - \frac{r - r_M}{\Delta} + \frac{t - t_F}{\Delta} V_M \right) , \qquad (45)$$

with the boundary condition $p(r_F) = p_a$. Integrating Eq. (24) over r from r_0 to r_M and using Eq. (43), we obtain

$$0 = \frac{dH}{dt} + H^2 , \qquad (46)$$

the solution of which is

$$H(t) = (H_0^{-1} + t - t_0)^{-1} . \qquad (47)$$

Integrating Eq. (24) over r from r_M to r_F and using Eqs. (43)-(45), we obtain

$$\frac{1}{m_a}(p_M - p_a) = \int_{r_M}^{r_F} dr \, N \left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial r} \right) = \frac{1}{12} V_M^2 (5N_a + N_M) , \qquad (48)$$

which is independent of the Δ appearing in Eqs. (44) and (45). We derive from Eq. (48), with the use of Eqs. (19) and (28), where we write r_M for $r_F = r_M + \Delta$ (in practice it is $r_F \simeq r_M$ since the thickness Δ of the wave front is extremely small)

$$V_M = \left\{ \frac{12k \left[2T(r_0/r_M)^2 - T_a \right]}{m_a [5 + 2(r_0/r_M)^2]} \right\}^{1/2} , \qquad (49)$$

where T is given by Eq. (42) and T_a is the ambient temperature.

For $r_M = r_0$, i.e., at the beginning of the expansion, it is $T_a \ll 2T$ so that Eq. (49) reduces to

$$V_M(r_0) \simeq \left[\frac{24kT(r_0)}{7\, m_a} \right]^{1/2} \simeq 1.6 \left(\frac{4kT}{3m_a} \right)^{1/2} = 1.6 V_s \qquad (50)$$

which is 1.6 times the speed of sound V_s at the initial temperature of the discharge. Equating Eq. (50) to Eq. (43) with the use of Eq. (47) for $t = t_0$, we find

$$H_0 = \frac{1}{r_0} \left[\frac{24kT(r_0)}{7\, m_a} \right]^{1/2} . \qquad (51)$$

Finally, we derive from Eqs. (43), (47), (49), and (51)

$$t(r_M) = t_0 - r_0 \left[\frac{7\,m_a}{24kT(r_0)}\right]^{1/2} + r_M \left\{\frac{m_a\left[5+2(r_0/r_M)^2\right]}{12k\left[2T(r_0/r_M)^2 - T_a\right]}\right\}^{1/2} , \quad (52)$$

where T is given by Eq. (42) with r_M for r_F.

Since $t(r_M)$ is contained in Eq. (42) for T, we should solve this equation that, after eliminating the radicals, turns out to be of the 5th degree and has therefore no analytical solution. We solve it numerically setting $r_M = r_g$ so as to find the time t_g at which the expanding discharge reaches the gap boundary. We obtain

$$\begin{aligned} t_g(C = 3.34\mu\mathrm{F}) &\simeq 0.392\mu\mathrm{s} = 0.0212\,[2\pi/\omega(C)] , \\ t_g(C = 6.68\mu\mathrm{F}) &\simeq 0.380\mu\mathrm{s} = 0.0151\,[2\pi/\omega(C)] , \\ t_g(C = 10.02\mu\mathrm{F}) &\simeq 0.376\mu\mathrm{s} = 0.0119\,[2\pi/\omega(C)] . \end{aligned} \quad (53)$$

After reaching the boundaries of the gaps, the expansion goes on but the discharge is only inside the gaps. We must therefore set $r_F = r_g$ in Eq. (34) thus deriving from it

$$T = P(t)^{1/4}\left[A\sigma_s\,(2\pi r_g l + \pi r_g^2)\right]^{-1/4} . \quad (54)$$

Substituting Eq. (54) into Eq. (49) with $r_M = r_g$, we obtain $V_M(r_g, t)$. Then from the continuity Eq. (26) in Eulerian form and with $r = r_g$, we derive

$$N(t) = N(r_g) \exp\left[-\frac{2}{r_g}\int_{t_g}^{t} V(r_g, t)\,dt\right] . \quad (55)$$

B. Air as a non-macroscopic plasma of ions and electrons

If the expanding gas could be considered as a macroscopic fluid, all the found quantities would decrease with increasing r_M, hence t, until the internal pressure equates the atmospheric pressure. However, as N decreases, the air can no longer be considered as a continuous "macroscopic fluid", but we must consider its atomic, or better ions and electrons, composition. The process becomes diffusive on the front of the expanding discharge. The pressure inside the gap does not reduce asymptotically to the atmosferic pressure p_a but, until $T \gg T_a = 300K$, there is a diffusion from outside to inside even if $p > p_a$. The flow density from inside to outside is unidirectional and given by NV_M. Equilibrium is reached when the outflow equates the inflow which has an isotropic distribution so that only the component against the outflow is effective. The balance is therefore

$$\begin{aligned} N(t^*)V_M(r_g) &= \int_0^{\pi/2} d\vartheta \frac{1}{2}\sin\vartheta N_a v_{300}\cos\vartheta \\ &= \frac{1}{4}N_a v_{300} = \frac{1}{4}N_a\sqrt{\frac{3k\,300}{m_a}} . \end{aligned} \quad (56)$$

C (μF)	t^* (μs)	$N(t^*)$ (m^{-3})	$T(t^*)$ (K)
3.34	0.589	6.40×10^{23}	4.16×10^4
6.68	0.544	6.25×10^{23}	4.08×10^4
10.02	0.539	6.22×10^{23}	4.03×10^4

TABLE II: Values of the concentrations N and of the temperatures T, in correspondence of the time t^* at which the regime is dominated by diffusion, for different capacitances C.

By means of Eqs.(49), (55), and (56) we obtain the times t^* at which the equilibrium expressed by Eq. (56) is reached, and the corresponding concentrations are reported in Table II.

A good interpolating expression for $t^*(C)$ is

$$t^*(C) = 0.674 - 0.21\,(C/6.68) + 0.08\,(C/6.68)^2 , \quad (57)$$

where C is measured in μF and t^* in μs.

The process would become steady-state if T were constant. Actually, T decreases because the electrical current in the circuit is a damped oscillation with decay time constant τ. Consequently, $N(r < r_g)$ starts increasing until it equates the external density when $T \to 300$ K.

To formulate what said we denote by $N_{\mathrm{in}}v_{\mathrm{in}}$ the diffusive inflow and $N_{\mathrm{out}}v_{\mathrm{out}}$ the outflow. The quasi steady-state condition implies, for any $r < r_g$ value,

$$N_{\mathrm{out}}(r)v_{\mathrm{out}}(r) + N_{\mathrm{in}}(r)v_{\mathrm{in}}(r) = 0. \quad (58)$$

where, from Eq. (56), we have

$$4N_{\mathrm{out}}(r_g)v_{\mathrm{out}}(r_g) \simeq N_a v_{300}. \quad (59)$$

For simplicity we assume v_{out} to be equal to the macroscopic velocity $V(r, t_g)$ of the previous phase of ordered expansion before the diffusive quasi-equilibrium was reached. We write therefore, by means of Eqs. (43) and (47), with $t = t_M \simeq t_g$ given by Eq. (53)

$$v_{\mathrm{out}} \simeq V = \frac{r}{t_g - t_0 + 1/H_0}. \quad (60)$$

Similarly, we take, using Eq. (28)

$$N_{\mathrm{out}} = 2N_a(r_0/r_g)^2 \quad (61)$$

where r_0 is given by Eqs. (37)-(41).

The inflow, being purely diffusive, is expressed by

$$N_{\mathrm{in}}(r)v_{\mathrm{in}}(r) = -D\frac{d}{dr}N_{\mathrm{in}}(r) . \quad (62)$$

Denoting $\lambda(v)$ the mean free path, the diffusion coefficient is given by

$$D = \frac{1}{3}\langle\lambda(v)\,v\rangle , \quad (63)$$

The entering molecules of air pratically do not interact with the much lighter electrons. The entering diatomic molecules of air split into two atoms but remain practically neutral. The entering neutral atoms therefore interact almost in the same way with inside neutrals and ions. It is therefore

$$\lambda(v) = \lambda = \frac{1}{\sigma_{nn} N_{\rm in}} = \frac{1}{\pi (2R_{at})^2} \frac{1}{N_{\rm in}} , \tag{64}$$

where $R_{at} = 6.4 \times 10^{-11}$ m is the wheighted average radius of an air atom (the empirical values of the atomic radii are: 65×10^{-12} m for nitrogen and 60×10^{-12} m for oxygen, respectively [4]). Substituting Eq. (64) into Eq. (63) and calculating v at the temperature $T(r_g)$ corresponding to the end of the ordered expansion (i.e., with $r_M \simeq r_g$), we obtain

$$D = \frac{< v >}{12\pi {R_{at}}^2 N_{\rm in}} = \frac{(2kT(r_g)/m)^{1/2}}{6\pi^2 {R_{at}}^2} \frac{1}{N_{\rm in}} = \frac{\overline{D}}{N_{\rm in}} \tag{65}$$

We derive from Eqs. (58), (60), (62), and (65)

$$\frac{\overline{D}}{N_{\rm in}} \frac{d}{dr} N_{\rm in}(r) = N_{\rm out} \frac{r}{t_g - t_0 + 1/H_0} . \tag{66}$$

Separating the variables and integrating both sides from r to r_g with $N_{\rm in}(r_g) = N_a - N_{\rm out}$, we obtain

$$\begin{aligned} N(r) &= N_{\rm in}(r) + N_{\rm out} = N_{\rm out} + (N_a - N_{\rm out}) \\ &\times \exp\left[N_{\rm out} \frac{r^2 - r_g^2}{2\overline{D}(t_g - t_0 + 1/H_0)} \right] , \end{aligned} \tag{67}$$

with $N_{\rm out}$ given by Eq. (61), $\overline{D}$ by Eq. (65), H_0 by Eq. (51), t_0 by Eqs. (39)-(41), and t_g by Eq. (53). The density $N(r)$ of Eq. (67) is the wanted expression.

IV. ELECTRONS AND IONS IN RUNAWAY AS THE CAUSE OF THE NET IMPULSE GIVEN TO THE CIRCUIT MOBILE SECTION OF GRANEAU EXPERIMENT

The preceding section was useful to find the concentration $N(r)$, the temperature T, and the time $t_M \simeq t_g$ necessary to reach the quasi-equilibrium condition due to back diffusion. However, during that phase the pressures p_1 and p_2 in the two gaps of Graneau *et al.* experiment [1] are equal so that there is no net impulse on the mobile armature (as denoted by the Authors in Ref. [1]). We outline here the mechanism that leads to a net impulse communicated to the armature.

When the quasi-equilibrium is reached, $N(r)$ is sufficiently low, while the temperature and the electric field are sufficiently high, to allow the production of runaways. An electron in runaway hitting an electrode can extract an average number n_{ee} of electrons and a number n_{ei} of ions. Similarly, an ion in runaway can extract n_{ii} ions and n_{ie} electrons. However, the ions extracted by an ion fall on the electrode after describing a short section of a parabola and are therefore not effectively contributing to the impulse transmitted to the armature. Consequently, n_{ii} does not matter. Similarly, for n_{ee}. What counts is n_{ei} and n_{ie}.

As far as we know, there are no experimental values for those two numbers in the extreme conditions of an electrical discharge. We therefore leave them as two unknowns to be determined by data fitting with the experimental results of Graneau *et al.* experiment [1].

The electrons and ions extracted from the electrode (because of the impinging on it of either an electron or an ion) have a distribution of velocities and the more energetic are already in runaway condition. Their trajectory is parabolic and the probability of impinging on the mobile armature is higher for the smaller gap. In the larger gap a good amount of electrons (or ions) in runaway can get out of the gap, so that they do not contribute to the impulse communicated to the armature. The successive extractions exalt the impulse differences to the armatures in the two gaps. The current in the gaps becomes mainly due to the runaways and the avalanche process is limited by the drop of the potential across the two gaps in series.

To formulate in a quantitive way the above process that lead to a pressure difference [expressed by Eq. (18)] we proceed as follows. We first find a plausible distribution for the velocities of the extracted electrons (or ions). Then we calculate the probability that an extracted electron (or ion) impinge on the armature. The point of extraction is generic and we have to perform an average. Finally, considering the very large number of successive extractions, we obtain a convergent geometric series that depends on the gap width (l_1 or l_2) and gives the differences on the impulses communicated to the armature in the two gaps.

The energy acquired by an electron in traversing a gap of height l_1 is eEl_1, the electric field E being equal in the two gaps in series. An energy balance yields

$$eEl_1 = \frac{1}{2} M u_0^2 + U_{\rm extr} + U_{\rm lost} , \tag{68}$$

where M and u_0 denote the mass and the speed, respectively, of an extracted ion, $U_{\rm extr}$ the extraction energy, and $U_{\rm lost}$ the energy transformed into thermal energy of the electrode. Now $U_{\rm extr}$ is of the order of some eV and is therefore negligible compared to $eEl_1 > 200$ eV even with the minimum used height $l_1 \simeq 1$ mm and at the end of the effective damped oscillation of the current in the circuit. $U_{\rm lost}$ is statistical and can range from almost zero to eEl_1. The maximum speed is therefore

$$u_{0\,\rm max} = \sqrt{2eEl_1/M} = \sqrt{2Al_1} . \tag{69}$$

A similar reasoning holds for the maximum speed $v_{0\,\rm max}$ of an electron extracted by an ion. It is therefore

$$v_{0\,\rm max} = \sqrt{2eEl_1/m} = \sqrt{2al_1} . \tag{70}$$

The distribution function $g_e(v_0, \theta)$ of the extracted electrons has statistically its maximum at $v_0 \simeq v_{0\,\max}/2$, being θ the angle between the initial velocity of the extracted electron and the outward normal to the electrode. A plausible renormalized distribution with this characteristics is

$$g_e(v_0, \theta) = \pi^3 \left[v_{0\,\max}^3 \left(1 - \frac{\mathcal{F}}{2}\right) (\pi^2 - 4) \right]^{-1} (1 - \mathcal{F} \cos\theta) \times \sin\left(\pi \frac{v_0}{v_{0\,\max}}\right) \Theta(v_{0\,\max} - v_0) \ , \quad (71)$$

where $\mathcal{F}$ is an unknown parameter that could be determined by data fittings, $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$.

A similar g_i holds for ions, substituting u for v.

Let us consider an axis x lying on one electrode and starting from the common axis of the gaps. Let an electron (or an ion) be extracted from a point of the x axis, distant r from the center and with an initial velocity

$$\mathbf{v}_0 = v_0(\hat{\mathbf{e}}_z \cos\theta + \hat{\mathbf{e}}_x \sin\theta\cos\phi + \hat{\mathbf{e}}_y \sin\theta\sin\phi) \ . \quad (72)$$

Neglecting the boundary effects, the acceleration of the considered electron (or ion) is simply expressed by

$$\mathbf{a} = \frac{e}{m} E \hat{\mathbf{e}}_z \quad (73)$$

so that after a time t from the extraction, its velocity is

$$\mathbf{v}(t) = \hat{\mathbf{e}}_z \left(v_0 \cos\theta + \frac{e}{m} E\, t\right) + v_0 \sin\theta (\hat{\mathbf{e}}_x \cos\phi + \hat{\mathbf{e}}_y \sin\phi) \ , \quad (74)$$

and its position is

$$\mathbf{R} = r\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z \left(v_0 t \cos\theta + \frac{e}{2m} E\, t^2\right) + v_0 t \sin\theta (\hat{\mathbf{e}}_x \cos\phi + \hat{\mathbf{e}}_y \sin\phi) \ . \quad (75)$$

The time Δt taken to reach the mobile armature is obtained projecting Eq. (75) on the z axis and putting $\mathbf{R} \cdot \hat{\mathbf{e}}_z = l_1$ (and, similarly, l_2 for the other gap). We obtain

$$\Delta t = -\frac{m}{eE} v_0 \cos\theta + \left[\left(\frac{m}{eE} v_0 \cos\theta\right)^2 + \frac{2ml_1}{eE} \right]^{1/2} \quad (76)$$

The condition that the electron impinges on the armature (rather than leaving the region of space of the gap) is obtained imposing that the component of $\mathbf{R}(\Delta t)$ on the xy plane has an absolute value less than r_g. It is

$$[r^2 + v_0^2 \Delta t^2 \sin^2\theta + 2 r v_0 \Delta t \sin\theta\cos\phi]^{1/2} \leq r_g \quad (77)$$

where Δt is given by Eq. (76). The solution of Eq. (77) with $=$ instead of $\leq$ and with Δt given by Eq. (76) leads to the initial, limiting velocity of capture v_{lc}. All the electrons, having initial velocity v_0 such that

$$v_0 \leq v_{lc}(r, \theta, \phi) \ , \quad (78)$$

hit the armature. Consequently, the average probability that an electron reaches the armature is given by

$$\mathcal{P}_e(t) = \int_0^{\pi/2} d\theta \sin\theta \int_0^{\pi} d\phi \frac{1}{\pi} \int_0^{r_g} dr\, r\, q(r) \times \int_{v_{0m}}^{v_{lc}(t,r,\theta,\phi)} dv_0\, g_e(v_0, \theta) \ , \quad (79)$$

where $g_e(v_0, \theta)$ is given by Eq. (71), $q(r)$ is the radial distribution of runaways, and v_{0m} the minimum value of the initial velocity (just after extraction from an electrode).

So far the procedure is exact if we neglect the stray fields due to the boundaries. However, to obtain v_{lc} we must solve Eq. (77) with Δt given by Eq. (76). Squaring twice to eliminate the radicals we obtain an equation of 8th degree. The numerical solution has to be obtained for every triad θ, ϕ, r in the four fold integral of Eq. (79). Taking into account that the above procedure should be repeated for all the values of the two gap heights used in the experiment of Ref. [1], we adopt an approximate procedure that implies a little loss of accuracy. We suppose that a fraction $\mathcal{F}$ of the initial velocities just after an extraction from an electrode be mainly along the z axis, while the complement $1 - \mathcal{F}$ be mainly in the xy plane. The fraction $\mathcal{F}$ can easily be found taking into account that the angular distribution of the extracted electrons is due to Coulomb scattering with the screening conditions studied by Brooks and Herring [5], whose differential cross-section is expressed by Eq. (A15) of Ref. [6] which reads

$$\sigma(v, \mu_s) = C\,[(1 - \mu_s) + D]^{-2} \ , \quad (80)$$

where μ_s is the cosine of the scattering angle ϑ_s and where C and D summarize quantities independent of the cosine μ_s of the scattering angle ϑ_s. Now D is important to eliminate the divergences of the Coulomb scattering at low ϑ_s angles, i.e, for $\mu_s \to 0$. However, in our case we consider angles $\vartheta_s > \pi/2$ so that D is negligible. We therefore approximate the electrons (or ions) extracted with initial velocities having scattering angles ϑ_s (with respect to the impinging electron, or ion) between $\pi/2$ and $3\pi/4$ as all moving in the xy plane (corresponding to $\vartheta_s = \pi/2$), while those with $3\pi/4 \leq \vartheta_s < \pi$ as moving along the z axis. Consequently, $1 - \mathcal{F}$ is proportional to the cross-section

$$\begin{aligned} 1 - \mathcal{F} &\propto \int_{\pi/2}^{3\pi/4} d\vartheta_s \sin\vartheta_s (1 - \mu_s)^{-2} \\ &= \int_{-2^{-1/2}}^{0} d\mu_s (1 - \mu_s)^{-2} \\ &= \left[(1 - \mu_s)^{-1}\right]_{-2^{-1/2}}^{0} = 0.4142 \ . \end{aligned} \quad (81)$$

Similarly

$$\begin{aligned} \mathcal{F} &\propto \int_{3\pi/4}^{\pi} d\vartheta_s \sin\vartheta_s (1 - \cos\vartheta_s)^{-2} \\ &= \left[(1 - \mu_s)^{-1}\right]_{-1}^{-2^{-1/2}} = 0.0858 \ . \end{aligned} \quad (82)$$

Since $1-\mathcal{F}+\mathcal{F}=1$, we find the constant of proportionality and obtain

$$1-\mathcal{F}=0.8284\ ; \qquad \mathcal{F}=0.1716\ . \tag{83}$$

Taking the same distribution for the velocities as given by Eq. (71) which accounts for multiple scatterings, we obtain the following normalized distribution function of the horizontal velocities v_0

$$g_{xy}(v_0)=\frac{0.8284\pi}{2v_{0\,\max}}\sin\left(\frac{\pi v_0}{v_{0\,\max}}\right)\Theta(v_{0\,\max}-v_0)\ . \tag{84}$$

Again with the use of Eq. (83), the distribution function for the velocities supposed along the z axis is given by

$$g_z(v_0)=\frac{0.1716\pi}{2v_{0\,\max}}\sin\left(\frac{\pi v_0}{v_{0\,\max}}\right)\Theta(v_{0\,\max}-v_0)\ . \tag{85}$$

We first calculate the probability that the fraction of the electrons with initial velocities parallel to the plane of the electrode (lying in the xy plane), reach the other face of the gap. For those electrons we can still neglect the stray fields due to the boundary, because the larger loss of those coming out radially is partially compensated by those pointing toward the center of the gap, while the radial stray fields have a negligible effect on the electrons having velocities roughly transversal to them. Consequently, still considering $\mathbf{E}=E\hat{\mathbf{e}}_z$, we can use the previous three-dimensional relationships simply putting $\theta=\pi/2$. Then the time Δt taken to reach the armature [and derived from Eq. (76)] is

$$\Delta t=(2ml_1/eE)^{1/2}\ . \tag{86}$$

Moreover, Eq. (77) reduces to

$$[r^2+v_0^2\Delta t^2+2rv_0\Delta t\cos\phi]^{1/2}\le r_g\ , \tag{87}$$

whence

$$v_0\le v_{lc}=\left(\frac{eE}{2ml_1}\right)^{1/2}[(r_g^2-r^2\sin^2\phi)^{1/2}-r\cos\phi]\ . \tag{88}$$

The average probability of hitting the armature is therefore given [even with the use of Eq. (84)]

$$\begin{aligned}\mathcal{P}_{e\,xy}(l,t) &= \int_0^{\pi}d\phi\frac{1}{\pi}\int_0^{r_g}dr\,r\,q(r)\int_{v_{0m}}^{v_{lc}(t,r,\phi)}dv_0 g_{xy}(v_0)\\ &= \frac{0.8284}{2\pi}\int_0^{\pi}d\phi\int_0^{r_{\max}}dr\,\frac{2r}{r_{\max}^2}\\ &\times\left\{\cos\left(\frac{\pi v_{0m}}{v_{0\,\max}}\right)-\cos\left[\frac{\pi v_{lc}(t,r,\phi)}{v_{0\,\max}}\right]\right\}\ ,\end{aligned} \tag{89}$$

which is a twofold integral instead of the fourthfold integral of Eq. (79). Moreover, v_{lc} needs not to be solved numerically for all the values of the variables in the points used for the numerical integration, because now we have its explicit analytical expression (88). As before, $v_{0\,\max}$ is given by Eq. (70).

A similar expression is obtained for the probability $\mathcal{P}_i(l,t)$ that an ion reaches the opposite electrode. Fortunately, the ratios of the velocities appearing in Eq. (89) and all the velocities turn out to be inversely proportional to the square root of the masses, so that

$$\mathcal{P}_i(l,t)=\mathcal{P}_{e\,xy}(l,t)=\mathcal{P}_{xy}(l,t)\ . \tag{90}$$

The dependence $v\propto m^{-1/2}$ is clear for $v_{0\,\max}$ and v_{lc} given by Eqs. (70) and (88), respectively.

Let us now calculate the probability $\mathcal{P}_z$ that either an ion (or an electron) having initial velocity $\mathbf{v}_0=v_0\hat{\mathbf{e}}_z$ reach the other face of the gap. If there were no stray, radial fields, all the fraction $\mathcal{F}$ [given by Eq. (83)] of these ions would reach the other face of the gap. In this case the stray, radial field E_r cannot be neglected. Since any gap face is equipotential, $E_r(z=0)=E_r(z=l)=0$, and also $E_r(l/2)=0$ because of symmetry, we can take a sinusoidal beahavior for it

$$\mathbf{E}_r(r,z,l_1)=\hat{\mathbf{r}}\,E_{r0}(r,l_1)\sin\left(\frac{2\pi z}{l_1}\right)\ , \tag{91}$$

where $\hat{\mathbf{r}}$ is the radial unit vector and $E_{r0}(0)=0$ because of symmetry. Consequently, we obtain

$$\begin{aligned}E_r(r,z,l_1) &= \frac{RI_0}{l\sqrt{2}}\exp\left(-\frac{t}{2\tau}\right)\frac{1.7l_1/l}{1+32(l_1/l)^2+6.5(l_1/l)^3}\\ &\times\sin\left(\frac{\pi r/r_g}{1+2.6l_1/l}\right)\sin\left(\frac{2\pi z}{l_1}\right)\ .\end{aligned} \tag{92}$$

Because of E_r, an ion undergoes a radial displacement given by

$$\Delta r=\int_0^{\Delta t}dt\int_0^t dt'\frac{e}{m}E_r(r,z,l_1)\ , \tag{93}$$

where Δt is the time taken by the ion to traverse the gap of length l_1 and given by Eq. (76) with $\theta=0$. The ion impinges on opposite face of the gap if

$$\Delta r<r_g-r\ . \tag{94}$$

As soon as an ion comes out of the cylinder of radius r_g, it finds the atmosferic concentration and loses the largest part of its momentum against the air molecules.

In order to perform the integration (93) we must express z as a function of the dummy variable t'. This is easily obtained if we keep constant $E_z=E\hat{\mathbf{e}}_z$ so that

$$z=v_0t'+\frac{e}{2m}Et'^2\ . \tag{95}$$

For simplicity, we keep $E_{r0}(r)$ equal to value corresponding to the starting r of the ion, and we perform numerical integration of Eq. (93) for different l_1, r, and v_0 (contained in Δt) values, thus obtaining, by means of interpolation,

$$\begin{aligned}\frac{v_0}{v_{0z_m}(l_1,r)} &= \left[-8.79\times10^{-3}+1.069\left(\frac{r}{r_g}\right)^2\right]\\ &\times\left[1.105-0.269\left(\frac{9\times10^{-3}-l_1}{l}\right)\right]\ .\end{aligned} \tag{96}$$

The runaway ions (or electrons) reaches the mobile armature only if $v_{0z} \geq v_{0z_m}(l,r)$. We therefore obtain for the probability $\mathcal{P}_z$ that an ion (or an electron) hits the armature face if its starting velocity is $\mathbf{v}_0 = v_0 \hat{\mathbf{e}}_z$, also using Eqs. (82), (83), and (96),

$$\mathcal{P}_z(l,t) = \frac{0.1716}{2\pi} \int_0^{r_{\max}} dr \, \frac{2r}{r_{\max}^2} \left\{ \cos\left[\frac{\pi v_{0z_m}(l,r)}{v_{0\,\max}}\right] + 1 \right\} , \tag{97}$$

similar to Eq. (89) but with $v_{0\,\max}$ [still given by Eq. (70)] for $v_{lc}(t,r,\phi)$ and no dependence on ϕ.

If n_{ei} ions are extracted from the armature by one electron, the number of runaway ions that reach the opposite electrode is $n_{ei}(\mathcal{P}_z + \mathcal{P}_{xy})$. Then these ions extract $n_{ie}n_{ei}(\mathcal{P}_{xy} + \mathcal{P}_z)$ electrons and, of these, the fraction

$$\begin{aligned} n_{\rm run}(l_1) &= n_{ei}n_{ie}[\mathcal{P}_z(l_1,t) + \mathcal{P}_{xy}(l_1,t)]^2 \\ &= \zeta[\mathcal{P}_z(l_1,t) + \mathcal{P}_{xy}(l_1,t)]^2 \end{aligned} \tag{98}$$

reaches the armature. The process repeats and, if the factor $n_{\rm run} < 1$, we have a convergent geometrical series so that the total number of electrons impinging on the armature and due to an initial electron in runaway is

$$n_{\rm tot}(l_1) = [1 - n_{\rm run}(l_1)]^{-1} . \tag{99}$$

Because of the symmetry in the subscripts i and e of Eq. (98), the same series holds for ions.

The net impulse communicated to an armature by an electron impinging on it is expressed by

$$\mathcal{I} = mv = \sqrt{2meEl_1}. \tag{100}$$

However, during the transit time, there is an attraction of the considered armature by part of the electron in flight causing a force $eE/2$ (the field due to a single armature is $E/2$).This reaction force produces an impulse with opposite sign with respect to Eq. (100) given by

$$\mathcal{I}_{re} = -\frac{1}{2}eE\Delta t = -\frac{eE}{2}\sqrt{\frac{2l_1}{a}} = -\frac{1}{2}\mathcal{I} . \tag{101}$$

The net impulse is therefore from Eqs. (100) and (101)

$$\mathcal{I}_{\rm net} = \mathcal{I} + \mathcal{I}_{re} = \frac{1}{2}\mathcal{I} = \sqrt{\frac{1}{2}meEl_1} . \tag{102}$$

An initial electron produces a total number $n_{\rm tot}(l_1)$ of runaways given by Eq. (99), so that the total net impulse is expressed by

$$\mathcal{I}_{\rm tot} = [1 - n_{\rm run}(l_1)]^{-1}\sqrt{\frac{1}{2}meEl_1} , \tag{103}$$

where $n_{\rm run}(l_1)$ is given by Eq. (98).

The net average force due to single initial electrons in runaway is given by $\mathcal{I}_{\rm net}$ times the frequency ν_b of the bounces on the considered armature. It is

$$\nu_b = (\Delta t_i + \Delta t_e)^{-1} = \left[\frac{2l_1}{eE}(\sqrt{m} + \sqrt{M})\right]^{-1} , \tag{104}$$

so that the net force on the armature on which electrons (due to a single initial electron in runaway) impinge is

$$F_e^1(l_1) = \mathcal{I}_{\rm tot}\nu_b = \frac{eE}{2[1 - n_{\rm run}(l_1)]}\frac{\sqrt{m}}{\sqrt{m} + \sqrt{M}} . \tag{105}$$

However, the electric field value E versus time is a damped oscillation so that E inverts its direction after only half-period and then ions, instead of electrons, impinge on the armature. Being the oscillations slowly damped, in first approximation we may take

$$F_{\rm net}^1(l_1) = \frac{1}{2}\left[F_e(l_1) + F_i(l_1)\right] = \frac{eE}{4[1 - n_{\rm run}(l_1)]} , \tag{106}$$

that is symmetric with respect to ions and electrons.

The total net force on the mobile armature is the difference of the forces on the two bases of the armature, i.e.,

$$\begin{aligned} F_{\rm net}^1 &= F_{\rm net}^1(l_1) - F_{\rm net}^1(l_2) \\ &= \frac{eE}{4}\left[\frac{1}{1 - n_{\rm run}(l_1)} - \frac{1}{1 - n_{\rm run}(l_2)}\right] . \end{aligned} \tag{107}$$

If $I_{\rm run} = dN_{\rm run}/dt$ is the total current of runaways, i.e. the number of electrons and ions in runaway per unit time expressed by

$$I_{\rm run} = \frac{2}{T}p_r f N_{\rm out} r_g(l_1 + l_2) , \tag{108}$$

where p_r is probability to have runaways as half the number of electrons with initial reduced velocities, f is given by Eq. (22), and $N_{\rm out}$ is obtained by Eq. (61), the total net impulse on the mobile armature is

$$\begin{aligned} \Gamma_{\rm tot} &= \int_{t^*}^{\infty} dt \, I_{\rm run}\frac{e}{4}E_0 \exp\left(-\frac{t}{2\tau}\right) \\ &\times \left[\frac{1}{1 - n_{\rm run}(l_1)} - \frac{1}{1 - n_{\rm run}(l_2)}\right] , \end{aligned} \tag{109}$$

where $t^*(C)$ is given by Eq. (57).

Finally, $M_A = 17$ g is the mass of the mobile armature [1], the maximum displacement or height h it would reach in absence of friction is

$$h = \Gamma_{\rm tot}^2(2M_A^2 g)^{-1} . \tag{110}$$

Our calculations are performed using the damped oscillations for the current in the circuit given in Ref. [1], which implies simmetry between ions and electrons. We suggest that other experiments of the kind of Graneau *et al.* but with an overdamped discharge, imply an asymmetry between $F_e^1(l_1)$ given by Eq. (106) and the corresponding for ions

$$F_i^1(l_1) = F_e^1(l_1)\sqrt{M/m} . \tag{111}$$

A dedicated experiment performed in this condition would be able to discriminate $F_e^1(l_1)$ and $F_i^1(l_1)$, for example, using first a strongly damped positive E, and repeating the experiment using a strongly damped negative

E. The outcome of such a repetition of the Graneau *et al.* experiment could be used as a test of our theoretical analysis that predicts that runaways are responsible for the force difference on the mobile armature.

V. DATA FITTING AND THE ITERATIVE PROCEDURE

As said in the Introduction, we apply our theoretical results to find the h values, i.e., the upward maximum displacements of the mobile armature for different values of the smaller gap l_1 and capacitance C. We use Eq. (110), where $\Gamma_{\rm tot}$ is given by Eq. (109), in which $I_{\rm run}$ is expressed by Eq. (108) and $n_{\rm run}(l)$ by Eq. (98), with $\mathcal{P}_{xy}(l,t)$ and $\mathcal{P}_z(l,t)$ given by Eqs. (89) and (97), respectively, the velocities $v_{0\,\rm max}$, v_{0m}, v_{lc}, and v_{0z_m} being expressed, in order, by Eqs. (70), (88), and (96). In our equations, there are three unknown parameters, namely ρ, $\zeta = n_{ei} n_{ie}$, and $\mathcal{T}$, which are important atomic quantities at the extreme temperatures and conditions of the electrical discharges, at present impossible to be obtained by other methods. Actually, ρ, appearing in Eq. (22), gives the contribution to the number f of ionized electrons per atoms due to the current I; $\zeta = n_{ei} n_{ie}$, appearing in Eq. (98), is the product of n_{ei} (number of ions extracted by one electron) and n_{ie} (number of electrons extracted by one ion); $\mathcal{T}$, appearing in Eq. (108), expresses the relaxation time at which the small range of speeds just before v_{0m} is practically reconstructed. Now a ρ value is necessary to calculate the time t^* [appearing in Eq. (109)] at which the process of the runaways starts. We have therefore adopted an iterative procedure putting, at zero order, $\rho = 0$ in Sec. III A, thus obtaining Eq. (57). It is with the t^* given by Eq. (57), and used in Eq. (109), that we have made a data fitting with the experimental data of Graneau *et al.* [1]. We find the correct dependence of h on the value l_1 of the smaller gap, and a good numerical equality to the experimental data provided the three sought parameters take the following first order values

$$\rho_1 = 1.46 \times 10^{-6}\ \text{A}^{-1}\ ; \quad \zeta_1 = 0.21\ ; \quad \mathcal{T}_1 = 88\ \mu\text{s}\ . \tag{112}$$

With this ρ_1 value in Eq. (22), and the same procedure of Sec. III A, we find the new t_2^* values that can be expressed by

$$t_2^*(C) \simeq 0.672 - 0.206(C/6.68) + 0.078(C/6.68)^2\ , \tag{113}$$

where C is measured in μF and t_2^* in μs.

We see that the t_2^* values given by Eq. (113) differ of only some percents from those given by Eq. (57). With the new t_2^*, the data fitting with the experimental data is shown in Table III, and the second order values of the three important atomic parameters are

$$\rho_2 = 1.41 \times 10^{-6}\ \text{A}^{-1}\ ; \quad \zeta_2 = 0.22\ ; \quad \mathcal{T}_2 = 86\ \mu\text{s}\ . \tag{114}$$

l_1 (mm)	C (μF)	t_2^* (μs)	$h_{\rm exp}$ (mm)	h (mm)
1.0	3.34	0.588	3	4.2
1.0	5.01	0.561	10.3	11.8
1.0	5.01	0.561	5.8	6.4
1.0	6.68	0.544	16.0	16.3
2.0	3.34	0.588	1.9	2.4
2.0	5.01	0.561	4.1	4.5
2.0	5.01	0.561	3.3	4.3
2.0	6.68	0.544	8.4	8.9
2.0	6.68	0.544	6.6	7.0
3.0	8.35	0.536	11.5	10.8
4.0	6.68	0.544	1.3	1.5
4.0	8.35	0.536	2.8	3.1
4.0	10.02	0.538	3.3	3.0
5.0	6.68	0.544	1.0	0.83
8.0	10.02	0.538	0.6	0.47
10.2			0.0	0.0

TABLE III: Comparison between the theoretical predictions vs experimental data for the height h of the mobile rod.

We see that these values little differ from those of Eq. (112), showing that our iterative procedure converges very rapidly.

The found $\mathcal{T}$ value deserves a comment. The usual relaxation time in the case of binary collisions between electrons and atoms (either neutral or ionized) is of the order $t_{\rm relax} = \nu^{-1}(v_m) M/(2m)$, where $\nu^{-1}(v_m)$ has to be taken in correspondance of roughly twice the minimum value v_{rm} to have runaways. With the value $v_{rm} = 3.02$, we derive

$$\nu^{-1}(2v_{rm}) \simeq 10^{-11}\ \text{s}. \tag{115}$$

Consequently, with $M/2m \simeq 1836 \times 7 \simeq 10^4$ we obtain

$$t_{\text{relax binary}} \simeq 10^{-7}\ \text{s}. \tag{116}$$

To have found the value given by Eq. (114) means that triple collisions are necessary to reconstruct the high velocity tail of the electron distribution function.

VI. CONCLUSIONS

It would have been practically impossible to conceive an experiment as the one performed by Graneau *et al.* [1] in order to derive the three atomic parameters ρ, $\zeta = n_{ei} n_{ie}$, and $\mathcal{T}$. It is a case of "heterogenesis of aims". Actually, Graneau *et al.* [1] devised their own experiment to validate Ampère's expression, but the correct, although rough, interpretation [2] of their effect, has on the contrary validated the standard formula of Grassmann. The point is that the much deeper (with respect to Ref. [2]) interpretation given in this paper included the above three parameters that has been determined by data fitting. Notice that the experimental points are 15, so that the agreement with the two main dependences, namely on the smaller gap length l_1, and on the maximum amplitude I_0 of the current (related to the value of the capacitance C), are a proof of the correctness of our interpretation. Let us clarify this point. The dependence of the maximum upward displacement h of the mobile armature on I_0 (or on C) is determined by the expression of the number f of extracted electron per atom. The general expression is rather easy to be devised because it must reduce to the thermal expression f_T for $I_0 \to 0$, and must be limited to 7 (number of electron of one atom of nitrogen) for $I \to \infty$. The wanted expression is Eq. (22) where an unknown parameter ρ appears. It is very difficult to find ρ theoretically so that we have left it as one of the three unknown parameters. The data fitting has given $\rho = 1.41 \times 10^{-6}$ A^{-1} (since ρI must give a number and I is measured in A, then the dimension of ρ is in A^{-1}).

The dependence of h on l_1 is correctly given by our theory based on the runaway electrons and ions. An unknown parameter contained in it is $\zeta = n_{ei} n_{ie}$ where n_{ei} (or n_{ie}) is the number of ions (or electrons) extracted from an electrode by one electron (or ion). In the extreme conditions of the electrical discharge there have been so far neither theory nor experiment able to determined it. Our data fitting has produced $\zeta = n_{ei} n_{ie} = 0.22$. In Sec. IV we have also suggested a modified version of the Graneau *et al.* [1] experiment by which it would be possible to separately derive n_{ei} and n_{ie}.

The third parameter we have obtained is the average time $\mathcal{T}$ of relaxation to reconstruct the high energy tail of the electron distribution function. We have found a single paper [7] that gives information in the case of cross-sections $\propto v^{-1}$ (hence constant collision frequency ν), and $\propto v^{-3}$ (hence $\nu \propto v^{-2}$), which are never found in any material in all the v range. Our data fitting have given $\mathcal{T} \simeq 8.6 \times 10^{-5}$ s [as given by Eq. (114)], i.e., a rather long time compared to the relaxation time for binary collisions which, in discharge conditions, is of the order 10^{-7} s, as given by Eq. (116). This is a sign that at least triple collisions are required to produce the high velocities in the distribution function. The $\mathcal{T}$ value is also required in the theory, we have just developed [8], that explains for the first time the long standing problem of the practically infinite memory of a fluctuation in the conduction current. The correlation function decays as $t^{-0.005}$ leading to a power spectral density of the fluctuating conductance G given by

$$S_G(\omega) = \frac{G^2 \alpha_\varepsilon}{2\pi \mathcal{N} \omega^{-0.995}} \,, \tag{117}$$

where $\mathcal{N}$ is the number of electrons between the sample electrodes, ω the angular frequency, and α_ε a coefficient that, differently from Hooge's empirical formula, depends on the electron concentration N. It is this dependence, together with the $\mathcal{T}$ value given in this paper, that leads to a close fitting with the experimental data.

We again encourage Graneau and collaborators to improve and modify their experiment whose correct interpretation can open a new stream of research.

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