The zero-point field of stochastic electrodynamics leads to a runaway for free electrons in semiconductors, thus explaining the $1/f$ noise

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The nonlinear Boltzmann equation with electron-electron scattering, reduced to an equivalent Fokker-Planck equation ($e-e$ FP), has a steady-state solution $f_0(v)$ (where v is the electron speed) which depends on a^2, where $\mathbf{a}$ is the acceleration due to the electric field $\mathbf{E}$. Including the $\mathbf{E}_{\rm ZPF}$ of the zero-point field of QED, $f_0(v)$ becomes similar to the Fermi-Dirac distribution function, and there is always a small δv range where the two collision frequencies ν_1 and ν_2 (appearing in the $e-e$ FP) are $\propto v^{-1}$, corresponding to the threshold of runaways. In that δv range, the time-dependent $f_0(v,\tau)$, hence the correlation function of the conductance G, decays as $\tau^{-\varepsilon}$ with $\varepsilon < 0.01$, leading to a power spectral density $S_G(f) = G^2\alpha_\varepsilon/(\mathcal{N} f^{1-\varepsilon})$, where f is the frequency, $\mathcal{N}$ the total number of electrons in the considered sample, and α_ε a dimensionless quantity that turns out to be in excellent agreement with the experimental data vs the electron concentration.

All the conceptual ideas for the solution of the so far unexplained $1/f$ (or flicker) noise has been given in the companion paper[1]. Here we formulate them in mathematical terms, and apply them to a sample of a semiconductor having length L between the two electrodes connected to the measuring instrument, in which a uniform current density j flows through a constant cross-section S under the action of a uniform electric field $\mathbf{E}$. The total current I flowing in the considered sample may be expressed as

$$I = jS = eNwS = \frac{e}{L}\mathcal{N}\mu_m E = GEL\ , \qquad (1)$$

where e is the electron charge, N the number concentration of the free electrons, $\mathcal{N} = NSL$ the total number in the considered sample, $w = \mu_m E$ the drift velocity, G the conductance, and μ_m the mobility given by

$$\mu_m = \frac{e}{m_*}\langle\mu(v)\rangle_f = \frac{e}{m_*}\int_0^\infty dv 4\pi v^2 f_0(v,t)\mu(v)\ , \qquad (2)$$

v being the electron speed, m_* the electron effective mass in the considered semiconductor (the introduction of m_* instead of the "free" electron mass m summarizes the effects of the electron interactions with the lattice), and $f_0(v,t)$ the isotropic component of the electron velocity distribution function which fluctuates as a function of time t. The conductance, in terms of microscopic quantities, is easily derived from Eqs. (1) and (2)

$$G(t) = \frac{e}{L^2}\mathcal{N}\mu_m = \frac{e^2}{m_* L^2}\mathcal{N}\langle\mu(v)\rangle_f\ . \qquad (3)$$

If we average over t or, since the process is ergodic, we take the ensemble average, we have $\langle G(t)\rangle = G$. The correlation function of $G(t)$ is the same of $g(t) = G(t) - G$, having zero mean value. It is

$$\begin{aligned} C_G(\tau) &= \langle g(t)g(t+\tau)\rangle = \left\langle G(t)G(t+\tau) - G^2\right\rangle \\ &= \left(\frac{4\pi e^2}{m_* L^2}\right)^2 \mathcal{N}\int_0^\infty dv v^2 f_0(v)\mu(v) \\ &\quad \times \int_0^\infty dv' v'^2\left[f_0(v',\tau|v) - f_0(v')\right]\mu(v'), \qquad (4) \end{aligned}$$

where $f_0(v',\tau|v)$ is the transition probability (or Green solution) to have the speed v' at time τ starting from v at $\tau = 0$. As experimentally and theoretically shown by Hooge[2], the $1/f$ noise is a fluctuaction of the mobility μ_m, hence of $G(t)$. The starting point to find the flicker noise is therefore the correlation given by Eq. (4). What we have to find in order to calculate it (hence the power spectral density given by Eq. (2) of Ref. 1) are the collision frequency $\nu_2(v)$ appearing in the reduced mobility $\mu(v)$ [see Eq. (2)], the isotropic, steady-state component $f_0(v)$ of the electron distribution function $f(v,\tau)$, and, above all, the transition probability $f_0(v',\tau|v)$.

Reduction of the nonlinear Boltzmann equation with electron-electron interactions to a Fokker-Planck equation and its solution in steady-state conditions

In a previous paper[3], the nonlinear Boltzmann equation with electron-electron ($e-e$) interaction has been

reduced to a Fokker-Planck equation ($e-e$ FP). The used method was partially analytical and partially numerical and the necessary use of modern computers explains why hundred years have been required for such an achievement. The important fact is that the result (the $e-e$ FP) is expressed in a compact analytical form.

In a subsequent paper[4], the method has been applied to doped silicon, and somewhat improved by exploiting axial symmetry and using quantum physics for the calculations of cross-sections (hence collision frequencies).

The resulting Fokker-Planck equation is given by Eq. (54) of Ref. 4 that we report here in a convenient version,

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^3\left[f_0(v,t)+\frac{kT}{m_* v}\frac{\partial f_0}{\partial v}\right]\nu_1(v)\right\} + \frac{a^2}{3\,v^2}\frac{\partial}{\partial v}\left[\frac{v^2}{\nu_2(v)}\frac{\partial f_0}{\partial v}\right], \tag{5}$$

where $f_0(v,t)$ is the isotropic component of the time dependent electron distribution function, v the absolute value of the electron velocity, $\mathbf{a} = e\mathbf{E}/m_*$ the electron acceleration due to an electric field $\mathbf{E}$, k the Boltzmann constant, and T the absolute temperature. Moreover, $\nu_1(v)$ is an equivalent collision frequency expressed as

$$\nu_1(v) = \frac{1}{3}A_0(v) + \frac{m}{M}\,\nu_m(v), \tag{6}$$

where M is an atom mass, ν_m the electron collision frequency for momentum transfer with ions and semiconductor lattice via acoustic and optical phonons (see Appendix A of Ref. 4), and $A_0(v)$ is expressed by Eq. (37) of Ref. 4 here reported

$$A_0(v) = \frac{Ne^4}{2\pi\epsilon^2\, m_*^2}\int_{-1}^{1} d\mu \int d^3\mathbf{v}'_*\, \overline{f}_0(v'_*,t)\,|\mathbf{v}-\mathbf{v}'_*|^{-3}\left\{\frac{m_*^2|\mathbf{v}'-\mathbf{v}'_*|^2}{\hbar^2\beta_s^2}\,\frac{1}{1+\hbar^2\beta_s^2\, m_*^{-2}|\mathbf{v}'-\mathbf{v}'_*|^{-2}}\right. \tag{7}$$
$$-\frac{m_*^2|\mathbf{v}-\mathbf{v}'_*|^2}{\hbar^2\beta_s^2} + \frac{1}{1+\hbar^2\beta_s^2\, m_*^{-2}|\mathbf{v}-\mathbf{v}'_*|^{-2}} - \frac{1}{1+2\hbar^2\beta_s^2\, m_*^{-2}|\mathbf{v}'-\mathbf{v}'_*|^{-2}}\,\ln\left(1+\frac{m_*^2|\mathbf{v}'-\mathbf{v}'_*|^2}{\hbar^2\beta_s^2}\right)$$
$$\left. + \frac{1}{1+2\hbar^2\beta_s^2\, m_*^{-2}|\mathbf{v}-\mathbf{v}'_*|^{-2}}\,\ln\left(1+\frac{m_*^2|\mathbf{v}-\mathbf{v}'_*|^2}{\hbar^2\beta_s^2}\right)\right\},$$

where, for the electron-electron scattering, $\mathbf{v}'$ and $\mathbf{v}'_*$ are the velocities of the colliding electrons immediately before a scattering event, $\mathbf{v}$ the electron velocity after scattering, $\mu = \mathbf{v}\cdot\mathbf{a}/(va)$, $\hbar$ the reduced Planck constant, β_s the inverse screening length, N the electron concentration, and ε the dielectric constant. Similarly,

$$\nu_2(v) = \langle\nu_{me}\rangle(v) + \nu_m(v), \tag{8}$$

where $\langle\nu_{me}\rangle(v)$ is the average value of the $e-e$ collision frequency for momentum transfer, given by Eq. (39) of Ref. 4, which reads

$$\langle\nu_{me}\rangle(v) = \frac{Ne^4}{2\pi\epsilon^2\, m_*^2}\int_{-1}^{1} d\mu \int d^3\mathbf{v}'_*\, \overline{f}_0(v'_*,t)\,|\mathbf{v}-\mathbf{v}'_*|^{-3}\left\{\frac{m_*^2|\mathbf{v}-\mathbf{v}'_*|^2}{\hbar^2\beta_s^2}\right. \tag{9}$$
$$\left. - \frac{1}{1+\hbar^2\beta_s^2\, m_*^{-2}|\mathbf{v}-\mathbf{v}'_*|^{-2}} - \frac{1}{1+2\hbar^2\beta_s^2\, m_*^{-2}|\mathbf{v}-\mathbf{v}'_*|^{-2}}\,\ln\left(1+\frac{m_*^2|\mathbf{v}-\mathbf{v}'_*|^2}{\hbar^2\beta_s^2}\right)\right\}.$$

Notice that two collision frequencies $\nu_1(v)$ and $\nu_2(v)$ appear in our $e-e$ FP given by Eq. (5), while a single $\nu(v)$ is present in the usual FP.

In steady-state conditions, i.e., for $\partial f_0(v,t)/\partial t = 0$, the solution of Eq. (5) is given by Eq. (58) of Ref. 4, which is a kind of Chapman-Cowling-Davydov expression

$$f_0(v) = Bp_0(v) = B\exp\int_0^v -\frac{m_* v\, dv}{kT + m_* a^2 (3\nu_1\nu_2)^{-1}}, \tag{10}$$

where B is the normalization constant and $p_0(v)$ the

N m^{-3}	a ms^{-2}	C	K^2 ms^{-2}	v_1 ms^{-1}	δv ms^{-1}
10^{20}	6.3×10^{18}	0.212	1.37×10^{19}	4.23×10^5	1.17×10^3
10^{21}	1.2×10^{20}	0.112	3.58×10^{20}	4.25×10^5	9.82×10^2
10^{22}	2.9×10^{21}	0.103	9.01×10^{21}	4.32×10^5	1.01×10^3
10^{23}	8.13×10^{22}	0.115	2.40×10^{23}	4.38×10^5	1.11×10^3
10^{24}	1.8×10^{24}	0.191	4.11×10^{24}	4.39×10^5	1.87×10^3
10^{25}	1.85×10^{25}	0.236	3.80×10^{25}	4.40×10^5	2.73×10^3
10^{26}	1.2×10^{26}	0.248	2.41×10^{26}	4.40×10^5	2.43×10^3

TABLE I: **Values of some fundamental parameters vs the concentration N of free electrons**. The electron acceleration a is mainly due to the ZPF, i.e., $a \simeq a_{\rm ZPF}$.

probability of occupation. The effect of the acceleration $\mathbf{a}$ is to produce an equivalent temperature $T_{\rm eq}$, or $\langle mv^2/2 \rangle = 3kT_{\rm eq}/2$. Since the square a^2 appears in both Eqs. (5) and (10), the effect of a high frequency oscillating field $\mathbf{E}$ is equivalent to a D.C. field provided we substitute the time average $\langle a^2 \rangle$ for a^2 in Eqs. (5) and (10).

Introduction of the zero-point field (ZPF) of QED

As discussed in Ref. 1, we do not neglect the ZPF of QED, thus writing $\mathbf{a} = \mathbf{a}_{D.C.} + \mathbf{a}_{ZPF}$, where, in the measurements of electrical noise, $\mathbf{a}_{D.C.}$ is the acceleration due to an external D.C. field and $\mathbf{a}_{ZPF}$ the acceleration due to the ZPF, whose power spectral density is given by Eq. (8) of Ref. 1. The a^2 appearing in Eq. (10) has therefore to be considered as a time average $a^2 = a^2_{D.C.} + a^2_{ZPF}$, being a_{ZPF} the effective value. We see below that while $a_{D.C.}$ is at maximum of the order 10^{13} m s^{-2}, at the frequencies of interest it is $a_{ZPF} > 10^{20}$ m s^{-2} so that $a^2 \simeq a^2_{ZPF}$. It is just with that high a value that the classical probability of occupation $p_0(v)$ can well describe the properties of a semiconductor. Actually, as shown in Ref. 1, we can obtain the Fermi energy U_F by means of the ZPF. What we do here, is to find a^2_{ZPF} such that Eq. (10) be as similar as possible to the Fermi-Dirac probability of occupation $p_{0F}(v)$. In a semiconductor, U_F is inside the forbidden gap. A common trick is to consider a parabolic conduction band starting from U_F, and to use a re-scaled $U_{F\,\rm res}$ put at the level of the parabolic conduction band. The compensation for this approximation is the introduction of the effective mass m_* for the electron mass m. A comparison of the two occupation probability is reported in Fig. 1, and the a^2 values (to have best agreement) vs N are reported in Table I. The important consequence is that, with those a^2 values, the two collision frequencies $\nu_1(v)$ and $\nu_2(v)$ involved in Eqs. (6)-(9) are inversely proportional to v in the same range $v_2 - v_1 = \delta v$. This behaviour is better shown in Fig. 2 where the products $\nu_1(v_r)v_r$ and $\nu_2(v_r)v_r$ are reported vs the reduced speed $v_r = v/\langle v^2 \rangle^{1/2}$. This behaviour, which is the threshold of runaways, may be formulated as

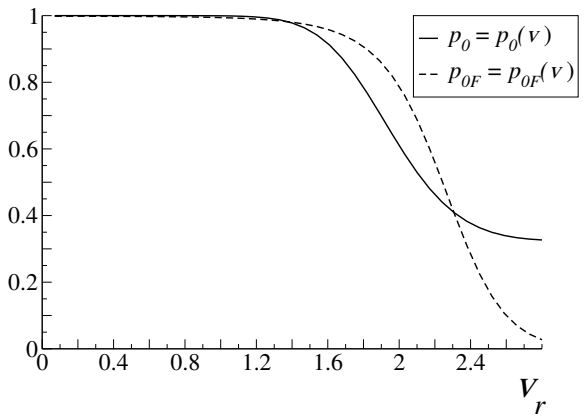

FIG. 1: **Comparison between the semi-classical probability of occupation $p_0(v)$ and the Fermi-Dirac probability of occupation $p_{0F}(v)$ vs the reduced velocity v_r.** It is $v_r = v/\langle v^2 \rangle^{1/2}$, where $\langle v^2 \rangle^{1/2} = 2.137 \times 10^5$ m s^{-1} for the concentration $N = 10^{22}$ m^{-3} and the acceleration $a = 2.9 \times 10^{21}$ m s^{-2}, essentially due to the zero-point-field (ZPF). The occupation probabilities are related to the corresponding distribution functions by $p_0(v) = f_0(v)/B$ and $p_{0F}(v) = f_{0F}(v)/B_F$, where the renormalization constants take the value $B_F \simeq B = 10^{-16}$ m^{-3} s^3 [it is $1 = \int_0^\infty dv 4\pi v^2 f_0(v)$].

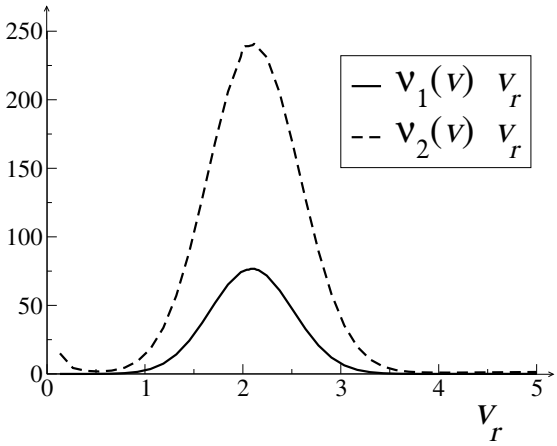

FIG. 2: **Plot of the functions $\nu_1(v_r)\,v_r$ and $\nu_2(v_r)\,v_r$ (in 10^{14} s^{-1}) vs v_r.** The reduced velocity v_r is defined in Fig. 1. In the interval $2.0215 < v_r < 2.0263$, the two functions are constant, so that the collision frequencies $\nu_1(v)$ and $\nu_2(v)$ go as $1/v$ satisfying Eq. (11).

$$\nu_2(v) = C\nu_1(v) = CK^2/v \ , \qquad (11)$$

where C and K^2 depend on N only, and we report their values in Table I, together with the beginning v_1 and the width δv of the effective speed interval.

Green solution of the $e-e$ FP, correlation function $\propto \tau^{-\varepsilon}$, power spectral density $S(f) \propto 1/f^{1-\varepsilon}$ in an indefinite semiconductor

In the effective interval δv, where Eq. (11) holds, our $e-e$ FP expressed by Eq. (5) reduces to the usual FP equation. Stenflo[5] was able to find a time dependent solution for the FP when Eq. (11) holds, but only when $T = 0$ K. In that case, Eq. (5) becomes

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left(K^2 v^2 f_0\right) + \frac{a^2}{3v^2}\frac{\partial}{\partial v}\left(\frac{v^3}{CK^2}\frac{\partial f_0}{\partial v}\right) \ . \quad (12)$$

The solution of this partial differential equation has been given by Stenflo[5] and reads

$$f_0(v,\tau_r) = \frac{Av^\alpha}{\tau_r}\exp\left(-\frac{v}{\tau_r}\right)\int_0^\infty du\, f_0(u,0)u^{-\alpha} \times \exp\left(-\frac{u}{\tau_r}\right) I_{2\alpha+4}\left(\frac{2\sqrt{vu}}{\tau_r}\right) \ , \quad (13)$$

where A is a constant to be determined by the initial conditions and I_p is the modified Bessel function of the first kind, and, using our parameters,

$$\alpha = -1 - \frac{3CK^4}{2a^2} \ ; \qquad \tau_r = \frac{ta^2}{3CK^2} \ , \quad (14)$$

C and K^2 being defined by Eq. (11), and $a^2 \simeq a^2_{\rm ZPF}$ (see the preceding section).

For large v and τ_r values, Eq. (13) reduces to[5]

$$f_0(v,\tau_r) = \frac{Av^{-3CK^4/a^2}}{\Gamma(\varepsilon)\tau_r^\varepsilon}\int_0^\infty du\, u^2\, f_0(u,0) \ , \quad (15)$$

where Γ is Euler's gamma function, and

$$\varepsilon = 2\alpha + 5 = 3(1 - CK^4/a^2) \ . \quad (16)$$

The ε values can in general be either positive, nil, or negative, tending to $-\infty$ for $a^2 \to 0$. However, and this is the most important consequence of not having neglected the ZPF, with $a^2 \simeq a^2_{\rm ZPF}$ it is $0.004 \le \varepsilon \le 0.006$, as shown in Table II. The time decay almost vanish for large v and τ_r values. On the other hand, the transition from the initial time decay expressed by Eq. (13) is always very slow so that the boundary conditions in the speed space have a very little influence. This fact is very important because Stenflo solved Eq. (12) considering Eq. (11) as valid in all the interval $0 \le v \le \infty$, while in our case it is satisfied in δv only and the boundary conditions in v space are different. Nevertheless, for very small ε values $f_0(v,\tau)$ depends on τ so weakly in δv that we have a crystallization of a fluctuation and we can have any boundary conditions because there is practically no time evolution.

In order to turn Eq. (15) into the transition probability $f_0(v',\tau|v)$ contained in the square bracket of Eq. (4), the initial distribution function $f_0(u,0)$ appearing in Eq. (15) has to be concentrated at a single v' value. Moreover, in order to be normalized $1 = \int_0^\infty du\, 4\pi u^2 f_0(u,0)$ it must take the expression

$$f_0(u,0) = (4\pi v'^2)^{-1}\,\delta(u - v') \ , \quad (17)$$

where δ denotes the Dirac's delta function.

Substituting Eq. (17) into Eq. (15), and using Eq. (14) with τ for τ_r, we obtain

$$f_0(v',\tau|v) = \frac{Av'^{\varepsilon-3}(3CK^2/a^2)^\varepsilon}{4\pi\Gamma(\varepsilon)\tau^\varepsilon} \ . \quad (18)$$

As said, Eq. (15), hence Eq. (18), is valid for sufficiently large v' and τ, i.e., for $v' > \langle v'^2\rangle^{1/2}$ and $\tau \gg t_f$, where $t_f \simeq \nu_2^{-1}(v')$ is a free flight time. Looking at Figs. 1 and 2 (relevant to $N = 10^{22}$ m^{-3}), we see that $v_1 \simeq 2.0215\langle v'^2\rangle^{1/2} = 4.32 \times 10^5$ ms^{-1} is larger than twice the square root of the mean square value. Moreover, the relaxation time τ_m of the high energy tail just in correspondence of v_1 turns out to be dominated by triple collisions and is of the order[6]

$$\tau_m \simeq 8.6 \times 10^{-5} s \ , \quad (19)$$

much greater than the average time of free flight $t_f \simeq \nu_2^{-1}(v) \simeq 8.6 \times 10^{-17}$ s. The two conditions $v' > \langle v'^2\rangle^{1/2}$ and $\tau \gg t_f$ are therefore well satisfied. In runaway conditions there is a process that turns the exponential decay into a power law, but τ_m remains the time of the transmission of information, and also the time necessary for the relaxation of Eq. (13) to Eq. (15).

Having found that Eq. (18) is valid for $\tau \gg \tau_m$, and that Eq. (13) is "crystallized" for $0 < \tau < \tau_m$, we may extend Eq. (18) to $\tau < \tau_m$ provided we use $\tau_m + \tau$ for τ. Then we can find the constant A, appearing in Eq. (18), by equating $f_0(v',0|v)$ to the equilibrium value $f_0(v')$ [given by Eq. (10)] in v_1 which is the beginning of the v interval δv where Eq. (11) holds. We obtain

$$f_0(v' = v_1, 0|v) = f_0(v_1) = Bp_0(v_1) = \frac{A(3CK^2 v_1/a^2)^\varepsilon}{4\pi v_1^3\Gamma(\varepsilon)\tau_m^\varepsilon} \ . \quad (20)$$

Splitting the second integral of Eq. (4) in three parts, from 0 to v_1, from v_1 to v_2, and from v_2 to ∞, the only part that brings about $1/f^{1-\varepsilon}$ noise is the second one. Noting that Eq. (18) vanishes for $\tau \to \infty$ (if $\varepsilon > 0$), so that $f_0(v') = 0$ for $v_1 < v' < v_2$, the only part of the correlation function that can lead to $1/f^{1-\varepsilon}$ noise is, with the use of Eqs. (4) and (18),

$$C_G^{\Delta v}(\tau) = \left(\frac{4\pi e^2}{m_* L^2}\right)^2 \mathcal{N}\int_0^\infty dv v^2 f_0(v)\mu(v)\int_{v_1}^{v_2}\frac{dv'}{4\pi} \times \frac{Av'^{\varepsilon-1}}{\Gamma(\varepsilon)(\tau_m+\tau)^\varepsilon}\left(\frac{3CK^2}{a^2}\right)^\varepsilon \mu(v') \ . \quad (21)$$

Again for $v_1 \le v' \le v_2$, we derive from Eqs. (2) and (11)

$$\mu(v') = \frac{4}{3}v'(CK^2)^{-1} \ . \quad (22)$$

Consequently, the second integral of Eq. (21) can easily be performed and does not depend on v. The first integral is therefore $\langle\mu(v)\rangle_f = B\langle\mu(v)\rangle_p$, the subscript "p" denotes that the average is over the probability of occupation $p_0(v)$, defined in Eq. (10). Then, Eq. (21) reduces to

$$C_G^{\Delta v}(\tau) = \left(\frac{e^2}{m_* L^2}\right)^2 \mathcal{N}\langle\mu(v)\rangle_f \frac{4A}{3\Gamma(\varepsilon)CK^2} \times \left[\frac{3CK^2}{a^2(\tau_m+\tau)}\right]^{\varepsilon} \frac{v_2^{1+\varepsilon} - v_1^{1+\varepsilon}}{1+\varepsilon} . \quad (23)$$

The power spectral density of $G(t)$ is the Fourier transform of its correlation and given by Eq. (2) of Ref. 1. Also using Eqs. (3) and (23) (to the aim of finding $1/f^{\varepsilon}$ only) writing $v_2 = v_1 + \delta v$ and performing a first order expansion, we obtain

$$\frac{S_G(f)}{G^2} = \frac{8A(3CK^2 v_1/a^2)^{\varepsilon}\delta v}{\mathcal{N}\langle\mu(v)\rangle_v 3\Gamma(\varepsilon)CK^2} \int_0^{+\infty} d\tau \frac{\cos(2\pi f\tau)}{(\tau_m+\tau)^{\varepsilon}} . \quad (24)$$

Setting $2\pi f\tau = x$ and $2\pi f\tau_m = x_m$, the integral of Eq. (24) becomes

$$\frac{1}{(2\pi f)^{1-\varepsilon}} \int_0^{+\infty} dx \frac{\cos x}{(x_m+x)^{\varepsilon}} = \frac{\mathcal{I}}{(2\pi)^{1-\varepsilon} f^{1-\varepsilon}} \quad (25)$$

Substituting Eqs. (20) and (25) into Eq. (24) and comparing the result with Eq. (5) of Ref. 1, we obtain

$$\alpha_{\varepsilon}(T=0) = \frac{32\pi v_1^3 p_0(v_1)\delta v(\tau_m f)^{\varepsilon}}{3(2\pi)^{1-\varepsilon}\langle\mu(v)\rangle_p CK^2}\mathcal{I} , \quad (26)$$

where it is pointed out that α_{ε} is predicted at zero temperature, since the calculations performed in Ref. 4 has been done at $T = 0$ to exploit Stenflo's result[5].

The numerical calculation of $\mathcal{I}$ implies reaching an $x_{\max}$ such that the denominator increases at least by a factor 100, i.e., up to $(x_m + x_{\max})^{\varepsilon} > 10^2 x_m^{\varepsilon}$, implying $x_{\max} > (10^{2/\varepsilon} - 1)x_m$. Since $x_m > 2 \times 10^{-10}$ and the maximum step of integration to keep a good accuracy is $2\pi/10$, the required number of points with the average value $\varepsilon = 0.005$, would be $> (5/\pi)2 \times 10^{-10} \times 10^{400} \simeq 3 \times 10^{390}$. It is possible to avoid this difficulty by performing an integration by parts taking $dx \cos x$ as the differential factor, so that

$$\mathcal{I} = \varepsilon \int_0^{+\infty} dx \frac{\sin x}{(x_m+x)^{1+\varepsilon}} . \quad (27)$$

The integral $\mathcal{I}/\varepsilon$ turns out to be practically insensitive to both ε (for $\varepsilon < 0.01$) and x_m (for $x_m < 10^{-3}$) and is equal to $\pi/2$. With this value, we derive from Eqs. (20) and (26)

$$\alpha_{\varepsilon}(T=0) = \frac{8\pi v_1^3 p_0(v_1)\delta v(2\pi\tau_m f)^{\varepsilon}\varepsilon}{3\langle\mu(v)\rangle_p CK^2} , \quad (28)$$

showing that the power spectral density of the noise is actually of the kind $1/f^{1-\varepsilon}$ with $0.004 \leq \varepsilon \leq 0.006$.

N m^{-3}	$10^3\varepsilon$	$p_0(v_1)$	$\langle\mu(v)\rangle_p$ $m^3 s^{-2}$	α_{ε}
10^{20}	5	0.55	3.97×10^4	1.74×10^{-5}
10^{21}	6	0.56	2.71×10^4	1.91×10^{-6}
10^{22}	6	0.53	1.15×10^4	2.00×10^{-7}
10^{23}	6	0.53	3.18×10^3	2.78×10^{-8}
10^{24}	5	0.49	3.25×10^2	1.24×10^{-8}
10^{25}	5	0.47	0.42×10^2	1.19×10^{-8}
10^{26}	4	0.46	0.45×10^1	1.16×10^{-8}

TABLE II: Values of other quantities complementary to those of Table I, and of the two fundamental results, i.e., the exponent ε of the decay time in Eq. (15) and the Hooge-like coefficient α_{ε}, vs the concentration N of free electrons.

Notice that the introduction of the probability $p_0(v)$ in Eq. (26) avoids the numerical calculations of a double integral to find the costant B in Eq. (10).

Since the collision frequency has been obtained numerically by means of Eqs. (8) and (9), it is better to perform an integration by parts of Eq. (2), thus obtaining Eq. (62) of Ref. 4. In our notations we have

$$\langle\mu(v)\rangle_p = -\frac{4}{3}\pi \int_0^{+\infty} dv \frac{v^3}{\nu_2(v)} \frac{\partial p_0(v)}{\partial v} , \quad (29)$$

which is a standard expression[7–9] containing the derivative of $f_0(v)$ analytically given by Eq. (10).

The values of the quantities appearing in Eq. (28) are reported in Tables I and II for seven different concentrations. In the latter there are the two fundamental results, i.e., the exponent ε of the time τ decay of the effective part of the correlation function (23), which leads to $f^{\varepsilon-1}$ of Eq. (25), and the parameter α_{ε} of the Hooge-like coefficient which, differently from Hooge's, depends on N. Since the results depend on τ_m and f, we have considered the two extreme cases, corresponding, for f, to the minimum and maximum measured values in the experiments considered by Hooge. It is $f_m \simeq 1$ s^{-1} and $f_M \simeq 10^4$ s^{-1} for "true" $1/f$ noise. In these two extremes, using Eq. (19), and with the maximum ε value, it is $(2\pi\tau_m f_m)^{\varepsilon} \simeq (2\pi 8.6 \times 10^{-5} \times 1)^{0.005} \simeq 0.96$, while $(2\pi\tau_m f_M)^{\varepsilon} \simeq (2\pi 8.6 \times 10^{-5} \times 10^4)^{0.005} = 1.01$. Consequently, we can write

$$(2\pi\tau_m f)^{\varepsilon} \simeq 0.98(1 \pm 0.1) . \quad (30)$$

The value used to obtain α_{ε} in Table II is the average value 0.98. The $\delta v = v_2 - v_1$ values are obtained taking v_1 and v_2 as the v values at which $\nu(v)v$ decreases by 2% with respect to the maximum value $[\nu(v)v]_M$ of the plateau. Actually, $\nu(v)v$ decreases very slowly in both sides of the $[\nu(v)v]_M$ up to $(1 - 0.02)[\nu(v)v]_M$, but then the decrease is rapid. The v_1 and δv values are taken from Table II. The results for different concentrations are reported in Table II and can be summarized as

$$\alpha_{\varepsilon}(T=0) = \frac{1.8 \times 10^{15}}{N + 10^{19}} \left(1 + \frac{N}{1.8 \times 10^{23}}\right)(1 \pm 0.1) . \quad (31)$$

This α_ε must not be compared to $\alpha_{\rm latt}$ given by Eq. (5) of Ref. 1, because the present theory shows that $1/f$ noise is almost exclusively given by electron-electron interactions (thus explaining its universality for the conduction current, independently of the material), and therefore it has nothing to do with a presumed interaction with the lattice. The comparison has to be done directly with the experimental results. In GaAs, the extrapolation for $T \to 0$ of the data of Fig.2 of Ref. 10 gives $\alpha_{\rm exp}(T = 0, N = 7 \times 10^{20}\ {\rm m}^{-3}) = 3 \times 10^{-6}(1 \pm 0.5)$; $\alpha_{\rm exp}(T = 0, N = 8 \times 10^{21}\ {\rm m}^{-3}) = 3 \times 10^{-7}(1 \pm 0.5)$; $\alpha_{\rm exp}(T = 0, N = 8 \times 10^{22}\ {\rm m}^{-3}) = 5 \times 10^{-8}(1 \pm 0.5)$. Both $\alpha_{\rm exp}$ and α_ε are reported in Fig. 1 of Ref. 1 vs N, and they are in agreement to within their uncertainties[11].

As shown at the end of Ref. 1, the power spectral density $S_G(f)$ given by Eq.(5) of Ref. 1, but with $\alpha_{\rm latt}$ substituted by α_ε given by Eq. (31), remains valid even for $f < w/L$, i.e., for times larger than the electron transit time L/w between the electrodes. The reason is that the memory of a fluctuaction is mainly due to $e-e$ interactions, and that the back diffusion is much more rapid than the drift velocity w.

The $1/f$ noise is therefore fully understood: our theory not only explains its origin but also predicts a dependence (so far unknown) of α_ε on the electron concentration N, which closely fits the experimental data.

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[1] Cavalleri, G. & Bosi, L. At last the origin of the $1/f$ noise has been understood. Submitted to *Nature (London) Phys.* as a companion paper.

[2] Hooge, F. N. $1/f$ noise sources. *IEEE Trans. Electron Devices* **41**, 1926-1935 (1994).

[3] Cavalleri, G. & Mauri, G. Approximate analytical solution of the nonlinear Boltzmann equation with an electron-electron interaction. *Phys. Rev. B* **49**, 9993-9996 (1994).

[4] Cavalleri, G., Tonni, E., Bosi, L. & Spavieri, G. Reduction of the non linear Boltzmann equation with electron-electron interaction to a Fokker-Planck equation and its steady-state solution for doped silicon. *Nuovo Cimento B* **116**, 1-30 (2001).

[5] Stenflo, L. Runaway in weakly ionized plasmas. *Plasma Phys.* **8**, 665-673 (1966).

[6] Cavalleri, G., Cesaroni, E., Tonni, E., & Spavieri, G. Atomic quantities derived from an electrodynamics experiment, *Phys. Rev. A* , submitted (2006).

[7] Gilardini, A. "*Low Energy Electron Collisions in Gases*" (J. Wiley, New York 1972), Eq. (2.3.27).

[8] Loeb, L. B. "*Recent Advances in Basic Processes of Gaseous Electronics*" (Berkeley University, California 1973).

[9] Huxley, L. J. H. & Crompton, R. W. "*The Diffusion and Drift of Electrons in Gases*" (Wiley, New York 1974).

[10] Ren, L. & Hooge, F. N. Temperature dependence of $1/f$ noise in epitaxial $n-$type GaAs. *Physica B* **176**, 209-212 (1992).

[11] The only available datum in Si at low T is $\alpha_{\rm exp}(T = 77K) \simeq 2 \times 10^{-7}$, as given by Clevers, R. H. M. $1/f$ noise in ion-implanted resistors between 77 and 300 KJ. *Appl. Phys.* **62**, 1877-1881 (1987). However, the sample has been doped with implantation of boron at $\simeq 1$ MeV, so that N is not uniform inside it.