The influence of local space anisotropy on the inertia of particles

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It is shown that in the case of spontaneous breaking of the original gauge symmetry, a dynamic rearrangement of vacuum may lead to the formation of an anisotropic fermion-antifermion condensate. The appearance of such a condensate creates a flat anisotropic, i.e. Finslerian event space. As a result, Minkowski geometry and, hence, Lorentz symmetry turn out to be violated. However relativistic symmetry of the event space remains valid. In the presence of the condensate it is represented by a 3-parameter group of the generalized Lorentz boosts. The principle of generalized Lorentz invariance makes it possible to take exactly into account the influence of local space anisotropy on the inertia of particles. In particular, it appeared that the inertial properties of a nonrelativistic particle in anisotropic space are specified by the tensor of inertial mass $m_{\alpha\beta}$. This tensor is obtained in an explicit form.

1 . Introduction

In spite of the impressive successes of the unified gauge theory of strong, weak and electromagnetic interactions, known as the Standard Model, one cannot a priori rule out the possibility that Lorentz symmetry underlying the theory is an approximate symmetry of nature. This implies that at the energies already attainable today empirical evidence may be obtained in favour of violation of Lorentz symmetry. At the same time it is obvious that the corresponding effects might manifest themselves only as strongly suppressed effects of Planck-scale physics.

According to the most popular point of view, even in a theory which has Lorentz invariance at the most fundamental level, this symmetry can be spontaneously broken if some (for example, vector) field acquires a vacuum expectation value which breaks the initial Lorentz symmetry. Certainly the question here concerns symmetry violation with respect to active Lorentz transformations of fundamental fields against the background of a fixed (in this case, vector) condensate. As for passive Lorentz transformations, under which the condensate is transformed as a Lorentz vector, the corresponding Lorentz covariance remains valid.

In order to describe possible effects caused by violation of active Lorentz invariance and to classify them as effects of Planck-scale physics, strongly suppressed at attainable energy scales, the so-called Standard Model Extension has been proposed [1-4]. The Lagrangian of this phenomenological theory is constructed so that it includes all passive Lorentz scalars formed by combining standard-model fields with coupling coefficients having Lorentz indices. As a result, along with Lorentz symmetry the relativistic symmetry turns out to be ad-hoc violated.

In contrast to such a string motivated theory (see also [5,6]) there exists another, Finslerian approach to the problem [7-18] which permits Lorentz symmetry violation without violation of relativistic symmetry. It is based on the following idea. Spontaneous breaking of the original gauge symmetry may be accompanied by the corresponding phase transitions in the geometrical structure of space-time. The point is that spontaneous breaking of gauge symmetry may lead to a dynamic rearrangement of vacuum which results in the formation of a relativistically invariant anisotropic fermion-antifermion condensate, i.e. of a constant classical nonscalar field. This constant field physically manifests itself as a relativistically invariant anisotropic medium filling space-time. Such a medium, leaving space-time flat, gives rise to its anisotropy, that is, instead of Minkowski space there appears a relativistically invariant Finslerian event space. Actually there appears either a flat space with partially broken 3D isotropy, i.e. axially symmetric Finslerian space or a flat Finslerian space with entirely broken 3D isotropy. These spaces together with their groups of isometries are described in sections 2 and 3. The equations of relativistic and nonrelativistic point mechanics generalized for the axially symmetric Finslerian space are discussed in section 4. The same equations but relating to entirely anisotropic Finslerian space are presented in section 5.

2 . Relativistically symmetric Finslerian space-time with partially broken 3D isotropy

In [7] the metric of the flat Finslerian space-time with partially broken 3D isotropy has been found in the form

$$ds^2 = \left[\frac{(dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2}{dx_0^2 - d\boldsymbol{x}^2}\right]^r (dx_0^2 - d\boldsymbol{x}^2)\,. \tag{1}$$

This metric depends on two constant parameters r and $\boldsymbol{\nu}$ and generalizes Minkowski metric. Instead of the 3-parameter group of rotations of Minkowski space, the space-time (1) admits only the 1-parameter group of rotations around the unit vector $\boldsymbol{\nu}$, which indicates a preferred direction in 3D space. No changes occur for translational symmetry: space-time translations leave the metric (1) invariant. As regards the transformations linking the various inertial frames, the usual Lorentz boosts modify the metric (1). Therefore, they do not belong to the isometry group of the space-time (1). By proper use of them, however, invariance transformations for the metric (1) can be constructed. The corresponding generalized Lorentz transformations (generalized Lorentz boosts) will be the following

$$x'^i = D(\boldsymbol{v},\boldsymbol{\nu})\, R^i_j(\boldsymbol{v},\boldsymbol{\nu})\, L^j_k(\boldsymbol{v})\, x^k\,, \tag{2}$$

where $\boldsymbol{v}$ is the velocity of the moving reference frame, $L^j_k(\boldsymbol{v})$ is the ordinary Lorentz boost, $R^i_j(\boldsymbol{v},\boldsymbol{\nu})$ is the further rotation of the spatial axes of the moving frame around the vector $[\boldsymbol{v}\boldsymbol{\nu}]$ through an angle

$$\varphi = \arccos\left\{1 - \frac{(1-\sqrt{1-\boldsymbol{v}^2/c^2})[\boldsymbol{v}\boldsymbol{\nu}]^2}{(1-\boldsymbol{v}\boldsymbol{\nu}/c)\boldsymbol{v}^2}\right\} \tag{3}$$

of relativistic aberration of $\boldsymbol{\nu}$, and $D(\boldsymbol{v},\boldsymbol{\nu})$ is a dilatational transformation of the space-time:

$$D(\boldsymbol{v},\boldsymbol{\nu}) = \left(\frac{1-\boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1-\boldsymbol{v}^2/c^2}}\right)^r I\,. \tag{4}$$

In the latter formula I is the unit matrix.

In contrast to Lorentz boosts, the generalized ones (2) make up a 3-parameter non-compact group with generators $X_1\,, X_2\,, X_3$. Thus, with the inclusion of the 1-parameter group of rotations around the preferred direction $\boldsymbol{\nu}$ and of the 4-parameter group of translations, the inhomogeneous group of isometries of the space (1) turns out to have 8 parameters. In order to obtain the simplest representation for its generators, it is sufficient to choose the third space axis along $\boldsymbol{\nu}$ and then to make use of the infinitesimal form of the transformations (2). As a result,

$$\begin{aligned}
&X_1 = -(x^1p_0 + x^0p_1) - (x^1p_3 - x^3p_1)\,,\\
&X_2 = -(x^2p_0 + x^0p_2) + (x^3p_2 - x^2p_3)\,,\\
&X_3 = -rx^ip_i - (x^3p_0 + x^0p_3)\,,\\
&R_3 = x^2p_1 - x^1p_2\,; \qquad\qquad p_i = \partial/\partial x^i\,.
\end{aligned} \tag{5}$$

The generators (5) satisfy the commutation relations

$$\begin{array}{llll}
[X_1X_2] = 0\,, & [R_3X_3] = 0\,, & & \\
{[X_3X_1]} = X_1\,, & [R_3X_1] = X_2\,, & & \\
{[X_3X_2]} = X_2\,, & [R_3X_2] = -X_1\,; & & \\
{[p_ip_j]} = 0\,; & & & \\
{[X_1p_0]} = p_1\,, & [X_2p_0] = p_2\,, & [X_3p_0] = rp_0 + p_3\,, & [R_3p_0] = 0\,,\\
{[X_1p_1]} = p_0 + p_3\,, & [X_2p_1] = 0\,, & [X_3p_1] = rp_1\,, & [R_3p_1] = p_2\,,\\
{[X_1p_2]} = 0\,, & [X_2p_2] = p_0 + p_3\,, & [X_3p_2] = rp_2\,, & [R_3p_2] = -p_1\,,\\
{[X_1p_3]} = -p_1\,, & [X_2p_3] = -p_2\,, & [X_3p_3] = rp_3 + p_0\,, & [R_3p_3] = 0\,.
\end{array} \tag{6}$$

From (6), we conclude in particular that the homogeneous isometry group of the space (1) contains 4 parameters (the generators $X_1\,, X_2\,, X_3\,, R_3$). Being a subgroup of the similitude group [19], it is isomorphic to the corresponding 4-parameter subgroup (with the generators $X_1\,, X_2\,, X_3\,|_{r=0}\,, R_3$) of the homogeneous Lorentz group. Since the 6-parameter homogeneous Lorentz group has no 5-parameter subgroup [20] while the 4-parameter subgroup is unique (up to isomorphisms), the transition from Minkowski space to the event space (1) implies a minimum of symmetry-breaking of the Lorentz symmetry, in which case the relativistic symmetry represented now by the generalized Lorentz boosts remains preserved.[1]

3. Relativistically symmetric Finslerian space-time with entirely broken 3D isotropy

In [13] and [14] the metric of the flat Finslerian space-time with entirely broken 3D isotropy has been found in the most general form:

$$\begin{aligned}
ds = {}&(dx_0 - dx_1 - dx_2 - dx_3)^{(1+r_1+r_2+r_3)/4}(dx_0 - dx_1 + dx_2 + dx_3)^{(1+r_1-r_2-r_3)/4}\\
&\times (dx_0 + dx_1 - dx_2 + dx_3)^{(1-r_1+r_2-r_3)/4}(dx_0 + dx_1 + dx_2 - dx_3)^{(1-r_1-r_2+r_3)/4}\,.
\end{aligned} \tag{7}$$

[1]Some examples of Finslerian spaces with a partial and entirely broken relativistic symmetry are considered in [21]-[24].

The anisotropy of the Finslerian space (7) is now specified by the three parameters $r_1\,,\ r_2\,,\ r_3$ which satisfy the conditions

$$\begin{array}{ll} 1+r_1+r_2+r_3 \geq 0\,, & 1+r_1-r_2-r_3 \geq 0\,, \\ 1-r_1+r_2-r_3 \geq 0\,, & 1-r_1-r_2+r_3 \geq 0\,. \end{array} \tag{8}$$

If we put $r_1=r_2=r_3=0$, we arrive at the Berwald-Moore metric

$$\begin{aligned} ds_{_{B-M}} = & \left[\,(dx_0-dx_1-dx_2-dx_3)(dx_0-dx_1+dx_2+dx_3)\right. \\ \times & \left.\;(dx_0+dx_1-dx_2+dx_3)(dx_0+dx_1+dx_2-dx_3)\,\right]^{1/4}. \end{aligned} \tag{9}$$

Thus, the metric (7) is the generalization of the Berwald-Moore metric (9).

The homogeneous 3-parameter noncompact isometry group, i.e. the relativistic symmetry group of the space-time (7) is Abelian. It can be verified that the transformations belonging to this group, i.e. the corresponding generalized Lorentz boosts have the form

$$x'_i = D\,L_{ik}\,x_k\,, \tag{10}$$

where

$$D = e^{-(\,r_1\,\alpha_1+r_2\,\alpha_2+r_3\,\alpha_3\,)}\,, \tag{11}$$

the matrices

$$L_{ik} = \begin{pmatrix} \mathcal{A} & -\mathcal{B} & -\mathcal{C} & -\mathcal{D} \\ -\mathcal{B} & \mathcal{A} & \mathcal{D} & \mathcal{C} \\ -\mathcal{C} & \mathcal{D} & \mathcal{A} & \mathcal{B} \\ -\mathcal{D} & \mathcal{C} & \mathcal{B} & \mathcal{A} \end{pmatrix} \tag{12}$$

are unimodular, in which case

$$\mathcal{A} = \cosh\alpha_1\cosh\alpha_2\cosh\alpha_3+\sinh\alpha_1\sinh\alpha_2\sinh\alpha_3\,,$$

$$\mathcal{B} = \cosh\alpha_1\sinh\alpha_2\sinh\alpha_3+\sinh\alpha_1\cosh\alpha_2\cosh\alpha_3\,,$$

$$\mathcal{C} = \cosh\alpha_1\sinh\alpha_2\cosh\alpha_3+\sinh\alpha_1\cosh\alpha_2\sinh\alpha_3\,,$$

$$\mathcal{D} = \cosh\alpha_1\cosh\alpha_2\sinh\alpha_3+\sinh\alpha_1\sinh\alpha_2\cosh\alpha_3\,,$$

$\alpha_1\,,\alpha_2\,,\alpha_3$ are the group parameters.

The transformations inverse to (10) can be obtained if we make the substitution

$$\alpha_1 \;\rightarrow\; -\alpha_1\,, \qquad \alpha_2 \;\rightarrow\; -\alpha_2\,, \qquad \alpha_3 \;\rightarrow\; -\alpha_3\,.$$

As a result

$$x_i = D^{-1}\,L^{-1}_{ik}\,x'_k\,, \tag{13}$$

where

$$L^{-1}_{ik} = \begin{pmatrix} \tilde{\mathcal{A}} & -\tilde{\mathcal{B}} & -\tilde{\mathcal{C}} & -\tilde{\mathcal{D}} \\ -\tilde{\mathcal{B}} & \tilde{\mathcal{A}} & \tilde{\mathcal{D}} & \tilde{\mathcal{C}} \\ -\tilde{\mathcal{C}} & \tilde{\mathcal{D}} & \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ -\tilde{\mathcal{D}} & \tilde{\mathcal{C}} & \tilde{\mathcal{B}} & \tilde{\mathcal{A}} \end{pmatrix}\,, \tag{14}$$

$$\tilde{\mathcal{A}} = \cosh\alpha_1\cosh\alpha_2\cosh\alpha_3-\sinh\alpha_1\sinh\alpha_2\sinh\alpha_3\,, \tag{15}$$

$$\tilde{\mathcal{B}} = \cosh\alpha_1\sinh\alpha_2\sinh\alpha_3-\sinh\alpha_1\cosh\alpha_2\cosh\alpha_3\,, \tag{16}$$

$$\tilde{\mathcal{C}} = \sinh\alpha_1 \cosh\alpha_2 \sinh\alpha_3 - \cosh\alpha_1 \sinh\alpha_2 \cosh\alpha_3 \,, \tag{17}$$

$$\tilde{\mathcal{D}} = \sinh\alpha_1 \sinh\alpha_2 \cosh\alpha_3 - \cosh\alpha_1 \cosh\alpha_2 \sinh\alpha_3 \,. \tag{18}$$

Since the relativistic symmetry transformations (10) have the same meaning as the Lorentz transformations, it is helpful to use as group parameters, in place of α_1, α_2, α_3, the components v_1, v_2, v_3 of the velocity of the primed frame. In order to obtain the necessary relations it is sufficient to put $x'_1 = x'_2 = x'_3 = 0$ in (13). As a result

$$v_1 = \frac{x_1}{x_0} = -\frac{\tilde{\mathcal{B}}}{\tilde{\mathcal{A}}}\,, \quad v_2 = \frac{x_2}{x_0} = -\frac{\tilde{\mathcal{C}}}{\tilde{\mathcal{A}}}\,, \quad v_3 = \frac{x_3}{x_0} = -\frac{\tilde{\mathcal{D}}}{\tilde{\mathcal{A}}}\,.$$

Taking into account (15–18), we can rewrite these formulas as follows

$$v_1 = (\tanh\alpha_1 - \tanh\alpha_2 \tanh\alpha_3)/(1 - \tanh\alpha_1 \tanh\alpha_2 \tanh\alpha_3)\,,$$

$$v_2 = (\tanh\alpha_2 - \tanh\alpha_1 \tanh\alpha_3)/(1 - \tanh\alpha_1 \tanh\alpha_2 \tanh\alpha_3)\,,$$

$$v_3 = (\tanh\alpha_3 - \tanh\alpha_1 \tanh\alpha_2)/(1 - \tanh\alpha_1 \tanh\alpha_2 \tanh\alpha_3)\,.$$

Now find the inverse relations, i.e. express α_1, α_2, α_3 in terms of v_1, v_2, v_3. This is easy to do if the following formulas are used

$$1 - v_1 - v_2 - v_3 = \frac{(1-\tanh\alpha_1)(1-\tanh\alpha_2)(1-\tanh\alpha_3)}{(1-\tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)}\,,$$

$$1 - v_1 + v_2 + v_3 = \frac{(1-\tanh\alpha_1)(1+\tanh\alpha_2)(1+\tanh\alpha_3)}{(1-\tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)}\,,$$

$$1 + v_1 - v_2 + v_3 = \frac{(1+\tanh\alpha_1)(1-\tanh\alpha_2)(1+\tanh\alpha_3)}{(1-\tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)}\,,$$

$$1 + v_1 + v_2 - v_3 = \frac{(1+\tanh\alpha_1)(1+\tanh\alpha_2)(1-\tanh\alpha_3)}{(1-\tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)}\,.$$

As a result we obtain

$$\alpha_1 = \frac{1}{4}\ln\frac{(1+v_1-v_2+v_3)(1+v_1+v_2-v_3)}{(1-v_1-v_2-v_3)(1-v_1+v_2+v_3)}\,,$$

$$\alpha_2 = \frac{1}{4}\ln\frac{(1-v_1+v_2+v_3)(1+v_1+v_2-v_3)}{(1-v_1-v_2-v_3)(1+v_1-v_2+v_3)}\,,$$

$$\alpha_3 = \frac{1}{4}\ln\frac{(1-v_1+v_2+v_3)(1+v_1-v_2+v_3)}{(1-v_1-v_2-v_3)(1+v_1+v_2-v_3)}\,.$$

As for the generators X_α of the homogeneous isometry group (10), they can be represented in the form

$$X_1 = -r_1 x_i\,\partial/\partial x_i - (x_1\,\partial/\partial x_0 + x_0\,\partial/\partial x_1) + (x_2\,\partial/\partial x_3 + x_3\,\partial/\partial x_2)\,,$$

$$X_2 = -r_2 x_i\,\partial/\partial x_i - (x_2\,\partial/\partial x_0 + x_0\,\partial/\partial x_2) + (x_1\,\partial/\partial x_3 + x_3\,\partial/\partial x_1)\,,$$

$$X_3 = -r_3 x_i\,\partial/\partial x_i - (x_3\,\partial/\partial x_0 + x_0\,\partial/\partial x_3) + (x_1\,\partial/\partial x_2 + x_2\,\partial/\partial x_1)\,.$$

It can easily be verified that these generators commute, i.e. $[X_\alpha X_\beta] = 0$.

Thus, with the inclusion of the 4-parameter group of translations, the inhomogeneous group of isometries of the space (7) turns out to have 7 parameters.

4. Generalization of the relativistic point mechanics for the Finslerian space-time with partially broken 3D isotropy

In order to generalize conventional relativistic point mechanics in accordance with the requirement of invariance of the corresponding equations under the group of generalized Lorentz boosts (2) it is sufficient in the action integral

$$S = -mc \int_a^b ds$$

to replace the pseudo-Euclidean expression for ds by the Finslerian expression (1). As a result, the Lagrangian function, corresponding to a free particle in the anisotropic space-time (1), takes the form

$$L = -mc^2 \left(\frac{1 - \boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \right)^r \sqrt{1 - \boldsymbol{v}^2/c^2}\,. \tag{19}$$

This Lagrangian function leads to the following expressions for the momentum $\boldsymbol{p}$ and the energy $E = cp^0$ of a relativistic particle

$$E = \frac{mc^2}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \left(\frac{1 - \boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \right)^r \left[1 - r + r\,\frac{1 - \boldsymbol{v}^2/c^2}{1 - \boldsymbol{v}\boldsymbol{\nu}/c} \right], \tag{20}$$

$$\boldsymbol{p} = \frac{m}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \left(\frac{1 - \boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \right)^r \left[(1 - r)\boldsymbol{v} + rc\boldsymbol{\nu}\,\frac{1 - \boldsymbol{v}^2/c^2}{1 - \boldsymbol{v}\boldsymbol{\nu}/c} \right]. \tag{21}$$

It can be verified by direct substitution that energy and momentum are related by the following equation of mass shell

$$\left[\frac{(p^0 - \boldsymbol{p}\boldsymbol{\nu})^2}{p^{0\,2} - \boldsymbol{p}^2} \right]^{-r} (p^{0\,2} - \boldsymbol{p}^2) = (mc)^2 (1 - r)^{(1-r)} (1 + r)^{(1+r)}\,. \tag{22}$$

This mass shell appears as a deformed two-sheeted hyperboloid inscribed into a cone $p^{0\,2} - \boldsymbol{p}^2 = 0$. For the upper sheet of such a "hyperboloid" p^0 reaches its absolute minimum $p^0_{min} = E_0/c = mc$ at $\boldsymbol{p} = \boldsymbol{p}_0 = rmc\boldsymbol{\nu}$. For the lower sheet, p^0 reaches its absolute maximum $p^0_{max} = -mc$ at $\boldsymbol{p} = -rm\boldsymbol{\nu}$.

In passing from one inertial frame to another the components $p^0 = E/c$ and $\boldsymbol{p}$ of the canonical 4-momentum p must transform such as to guarantee invariance of the form (22). We have noted in section 2 that the invariance of the Finslerian metric (1) is established by the generalized Lorentz boosts (2). From the comparison of (22) and (1), the invariance of (22) results from the transformations

$$p'^i = D^{-1}\, R^i_j\, L^j_k\, p^k\,, \tag{23}$$

where the matrices L_k^j and R_j^i are the same as in (2), while

$$D^{-1} = \left(\frac{1 - \boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \right)^{-r} I \,. \tag{24}$$

Thus, under the generalized Lorentz boosts the scale transformation (24) for momenta is inverse to the corresponding scale transformation (4) for the coordinates of events. Consequently, the phase of a plane wave is an invariant of the generalized Lorentz boosts.

Eq. (20) determines the dependence of the energy E of a free particle, present in the anisotropic space (1), on both the magnitude and the direction of its velocity $\boldsymbol{v}$. At $\boldsymbol{v} = 0$ the energy reaches its absolute minimum $E_0 = mc^2$. As regards the momentum $\boldsymbol{p}$, its direction, according to (21), does not coincide with the direction of the velocity of a massive particle. Even in the case $\boldsymbol{v} = 0$, the momentum of a particle does not vanish; there remains a "rest momentum" $\boldsymbol{p}_0 = rmc\boldsymbol{\nu}$. Massless particles have no such property; for them, as in conventional special theory of relativity, $v = c$ and $E^2/c^2 - \boldsymbol{p}^2 = 0$.

Being an intrinsic property of space, anisotropy is independent of the magnitude of relative velocities. Therefore, also nonrelativistic mechanics as a whole is different from the Newtonian case. In fact, in the nonrelativistic limit the following expressions are obtained from (20) and (21):

$$E = mc^2 + (1-r)\frac{m\boldsymbol{v}^2}{2} + r(1-r)\frac{m(\boldsymbol{v}\boldsymbol{\nu})^2}{2} \,, \tag{25}$$

$$\boldsymbol{p} = rmc\boldsymbol{\nu} + (1-r)m\boldsymbol{v} + r(1-r)m(\boldsymbol{v}\boldsymbol{\nu})\boldsymbol{\nu} \,. \tag{26}$$

Since within the framework of nonrelativistic mechanics the rest mass m is an additive quantity, the occurrence of the constant terms mc^2 and $rmc\boldsymbol{\nu}$ in (25) and (26) does not affect the conservation laws and the equations of motion. As a result, these terms can be omitted, and the kinetic energy and kinetic momentum, read off from (25) and (26), are

$$T = \frac{1}{2} m_{\alpha\beta}\, v^\alpha v^\beta \,, \qquad p_\alpha = m_{\alpha\beta}\, v^\beta \,, \tag{27}$$

where

$$m_{\alpha\beta} = m(1-r)(\delta_{\alpha\beta} + r\nu_\alpha\, \nu_\beta) \,. \tag{28}$$

Respectively, Newton's second law in anisotropic space takes the form

$$m_{\alpha\beta}\, a^\beta = F_\alpha \,; \qquad (\alpha = 1, 2, 3) \,. \tag{29}$$

As we see the inertial properties of a nonrelativistic particle in the anisotropic space (1) are specified by the tensor of inertial mass (28).

5. Generalization of the relativistic point mechanics for the Finslerian space-time with entirely broken 3D isotropy

Proceeding from the considerations of relativistic invariance and minimality on the straight world line, we write an action for a free particle in the entirely anisotropic space-time as

$$S = -mc \int_a^b ds \,,$$

where ds has the form (7). Then, for the action variation, we find the expession

$$\begin{aligned}\delta S =& -\int_a^b (p_0 d\delta x_0 - p_1 d\delta x_1 - p_2 d\delta x_2 - p_3 d\delta x_3)\\ =& (-p_0\delta x_0 + p_1\delta x_1 + p_2\delta x_2 + p_3\delta x_3)|_a^b\\ &+ \int_a^b [(dp_0/ds)\delta x_0 - (dp_1/ds)\delta x_1 - (dp_2/ds)\delta x_2 - (dp_3/ds)\delta x_3] ds\,.\end{aligned} \tag{30}$$

If we vary the trajectory under the condition that $(\delta x_i)|_a = (\delta x_i)|_b = 0$, then the principle of least action yields $p_i = const.$, i.e. rectilinear inertial motion. If we vary the coordinates of point b under the condition that $p_i = const.$, we get that

$$p_0 = -\frac{\partial S}{\partial x_0}\,, \quad p_\alpha = \frac{\partial S}{\partial x_\alpha}\,; \qquad \alpha = 1\,,2\,,3. \tag{31}$$

Thus, p_i is a canonical 4-momentum. In terms of 3-velocity, $v_\alpha = dx_\alpha/dx_0$, its components have the form

$$\begin{aligned}p_0 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\, & 1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3\\ &+ r_1[(1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3]\\ &+ r_2[(1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3]\\ &+ r_3[(1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2]\}\,,\end{aligned} \tag{32}$$

$$\begin{aligned}p_1 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\, & (1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3\\ &+ r_1[1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3]\\ &+ r_2[(1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2]\\ &+ r_3[(1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3]\}\,,\end{aligned} \tag{33}$$

$$\begin{aligned}p_2 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\, & (1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3\\ &+ r_1[(1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2]\\ &+ r_2[1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3]\\ &+ r_3[(1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3]\}\,,\end{aligned} \tag{34}$$

$$\begin{aligned}p_3 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\, & (1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2\\ &+ r_1[(1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3]\\ &+ r_2[(1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3]\\ &+ r_3[1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3]\ \}\,,\end{aligned} \tag{35}$$

where

$$\begin{aligned}(dx_0/ds)\left(ds_{B-M}/dx_0\right)^4 =& (1 - v_1 - v_2 - v_3)^{(3-r_1-r_2-r_3)/4}\\ &\times (1 - v_1 + v_2 + v_3)^{(3-r_1+r_2+r_3)/4}\\ &\times (1 + v_1 - v_2 + v_3)^{(3+r_1-r_2+r_3)/4}\\ &\times (1 + v_1 + v_2 - v_3)^{(3+r_1+r_2-r_3)/4}\,,\end{aligned} \tag{36}$$

in which case ds is the metric (7) and ds_{B-M} is the Berwald-Moore metric (9). Note that, beginning from eq. (32), we put $m = c = 1$.

There are points of similarity between the behaviour of a particle in the space (1) and in the space (7). For example, in accordance with (32) the energy p_0 of a free particle, present in entirely anisotropic space (7), also depends on the direction of its velocity v_α. At $v_\alpha = 0$ the energy reaches its absolute minimum $p_0 = 1$. As regards the momentum p_α, its direction, according to (33)-(35), does not coincide with the direction of the velocity of a massive particle. Even in the case $v_\alpha = 0$, the momentum of the particle does not vanish; there also remains a "rest momentum" $p_1 = r_1$, $p_2 = r_2$, $p_3 = r_3$. In addition, the inertial properties of a nonrelativistic particle in the entirely anisotropic space (7) are also specified by a tensor of inertial mass (see below).

In order to express the v_α in terms of the 4-momentum p_i it is sufficient (see [25]) to use eqs. (32)-(35). As a result we arrive at the relations

$$a_{\gamma\alpha} v_\alpha = b_\gamma\ , \tag{37}$$

where

$$\begin{aligned}
a_{11} = \quad a_{12} &= (p_0+p_3)(1+r_3) - (p_1+p_2)(r_1+r_2)\,,\\
a_{13} = \quad b_1 \ &= (p_1+p_2)(1+r_3) - (p_0+p_3)(r_2+r_3)\,,\\
a_{21} = -\, b_2 \ &= (p_0-p_1)(r_2-r_3) - (p_2-p_3)(1-r_1)\,,\\
a_{22} = -\, a_{23} &= (p_0-p_1)(1-r_1) - (p_2-p_3)(r_2-r_3)\,,\\
a_{31} = \quad b_3 \ &= (p_2+p_3)(1+r_1) - (p_0+p_1)(r_2+r_3)\,,\\
a_{32} = \quad a_{33} &= (p_0+p_1)(1+r_1) - (p_2+p_3)(r_2+r_3)\,.
\end{aligned}$$

At $r_1 = r_2 = r_3 = 0$, i.e. in the case of the Berwald-Moore space, eqs. (37) take the form

$$\begin{aligned}
(p_0+p_3)v_1 + (p_0+p_3)v_2 + (p_1+p_2)v_3 &= (p_1+p_2)\,,\\
(p_3-p_2)v_1 + (p_0-p_1)v_2 + (p_1-p_0)v_3 &= (p_2-p_3)\,,\\
(p_2+p_3)v_1 + (p_0+p_1)v_2 + (p_0+p_1)v_3 &= (p_2+p_3)
\end{aligned}$$

and lead to the following relations:

$$v_1 = \frac{p_1({p_0}^2 - {p_1}^2 + {p_2}^2 + {p_3}^2) - 2p_0p_2p_3}{p_0({p_0}^2 - {p_1}^2 - {p_2}^2 - {p_3}^2) + 2p_1p_2p_3}\,,$$

$$v_2 = \frac{p_2({p_0}^2 + {p_1}^2 - {p_2}^2 + {p_3}^2) - 2p_0p_1p_3}{p_0({p_0}^2 - {p_1}^2 - {p_2}^2 - {p_3}^2) + 2p_1p_2p_3}\,,$$

$$v_3 = \frac{p_3({p_0}^2 + {p_1}^2 + {p_2}^2 - {p_3}^2) - 2p_0p_1p_2}{p_0({p_0}^2 - {p_1}^2 - {p_2}^2 - {p_3}^2) + 2p_1p_2p_3}\,.$$

Proceeding from eqs. (32)-(35) as from the equations of mass shell given in a parametric form with the parameters v_α, we arrive at the algebraic form of the mass shell equation :

$$\left(\frac{p_0+p_1+p_2+p_3}{1+r_1+r_2+r_3}\right)^{(1+r_1+r_2+r_3)}\left(\frac{p_0+p_1-p_2-p_3}{1+r_1-r_2-r_3}\right)^{(1+r_1-r_2-r_3)}$$
$$\times\left(\frac{p_0-p_1+p_2-p_3}{1-r_1+r_2-r_3}\right)^{(1-r_1+r_2-r_3)}\left(\frac{p_0-p_1-p_2+p_3}{1-r_1-r_2+r_3}\right)^{(1-r_1-r_2+r_3)}=1. \qquad (38)$$

Let us consider a group of relativistic symmetry of the entirely anisotropic 4-momentum space and show that the transformations of 4-momenta, making up such a group, leave the mass shell equation (38) invariant. It is clear that the relativistic symmetry transformations of the entirely anisotropic momentum space are induced by the corresponding transformations (10) of the entirely anisotropic event space (7). In order to construct in an explicit form the linear transformations of 4-momenta, representing the relativistic symmetry group, we shall proceed from the definition of the canonical 4-momentum (31).

By virtue of (31) and (13),

$$p_0'=-\frac{\partial S}{\partial x_i}\frac{\partial x_i}{\partial x_0'}=D^{-1}(\,L^{-1}_{00}\,p_0-L^{-1}_{0\beta}\,p_\beta\,)\,, \qquad (39)$$

$$p_\alpha'=\frac{\partial S}{\partial x_i}\frac{\partial x_i}{\partial x_\alpha'}=D^{-1}(\,-L^{-1}_{\alpha 0}\,p_0+L^{-1}_{\alpha\beta}\,p_\beta\,)\,. \qquad (40)$$

Now, taking into account (14) we can unite (39) and (40). As a result,

$$p_i'=D^{-1}\,\mathcal{L}_{ik}\,p_k\,, \qquad (41)$$

where, owing to (11) and (14),

$$D^{-1}=e^{(\,r_1\,\alpha_1+r_2\,\alpha_2+r_3\,\alpha_3\,)}\,, \qquad (42)$$

$$\mathcal{L}_{ik}=\begin{pmatrix}\tilde{\mathcal{A}} & \tilde{\mathcal{B}} & \tilde{\mathcal{C}} & \tilde{\mathcal{D}}\\ \tilde{\mathcal{B}} & \tilde{\mathcal{A}} & \tilde{\mathcal{D}} & \tilde{\mathcal{C}}\\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} & \tilde{\mathcal{A}} & \tilde{\mathcal{B}}\\ \tilde{\mathcal{D}} & \tilde{\mathcal{C}} & \tilde{\mathcal{B}} & \tilde{\mathcal{A}}\end{pmatrix}\,, \qquad (43)$$

in which case α_1 , α_2 , α_3 are the group parameters and the martix elements $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$ are determined by eqs. (15)–(18). It is easy to verify that the linear transformations of 4-momenta (41) leave the mass shell equation (38) invariant and make up Abelian 3-parameter group of relativistic symmetry of the entirely anisotropic 4-momentum space.

At last, discuss briefly the inertial properties of a nonrelativistic particle in the entirely anisotropic space-time (7). The nonrelativistic motion in such a space is described by a nonrelativistic Lagrangian function or by a nonrelativistic Hamilton-Jacobi equation, which can be found from the corresponding relativistic expressions via the passage to the limit ($v/c\to 0$ or $c\to\infty$). Realizing this passage to the limit, one should bear it in mind that in the Lagrangian function there will be terms which constitute a total time derivative. These terms do not affect the equations of motion and can be omitted. Accordingly, in the action S we have to single out the terms which give the rest energy and the rest momentum. In addition note that the initial expression for relativistic Lagrangian function follows from the entirely anisotropic Finslerian metric (7). As regards the initial relativistically invariant Hamilton-Jacobi equation, it can be obtained by substituting the derivatives (31) into the mass shell equation (38). Taking into account these remarks and executing cumbersome calculations we get nonrelativistic tensor of inertial mass in the explicit form:

$$m_{\alpha\beta} = \left(\begin{array}{ccc} (1-{r_1}^2) & (r_3 - r_1 r_2) & (r_2 - r_1 r_3) \\ (r_3 - r_1 r_2) & (1-{r_2}^2) & (r_1 - r_2 r_3) \\ (r_2 - r_1 r_3) & (r_1 - r_2 r_3) & (1-{r_3}^2) \end{array} \right) . \tag{44}$$

Now, comparing (44) with (7) we see that the three parameters r_1, r_2, r_3 determine both the anisotropy of the flat Finslerian event space whose 3-rotational symmetry is entirely broken and the anisotropy of the inertia of a nonrelativistic particle present in such a space, i.e. the tensor of its inertial mass (44).

5. Conclusion

Having considered the relativistically invariant Finslerian spaces with partially and entirely broken 3D isotropy, we have generalized for such spaces the equations of conventional relativistic point mechanics. It has been shown that the lacal anisotropy of event space leads to the modification of Newton's second law. This means that the inertial properties of a nonrelativistic particle present in anisotropic space are specified by a tensor of inertial mass $m_{\alpha\beta}$. In the axially symmetric Finslerian space (1) such a tensor has the form of an axially symmetric tensor (28), while in the entirely anisotropic Finslerian space (7) it turns out to be the entirely anisotropic tensor (44). In other words the motion of a nonrelativistic particle in the axially symmetric anisotropic space is analogous to the motion of a quasiparticle in an axially symmetric crystalline medium, while its motion in the entirely anisotropic space is analogous to the motion of a quasiparticle in an entirely anisotropic crystalline medium. The role of the axially symmetric medium, which fills space-time and generates its partial anisotropy, is played by the axially symmetric relativistically invariant fermion-antifermion condensate, while the role of the entirely anisotropic medium is played by the entirely anisotropic relativistically invariant fermion-antifermion condensate.

In the case of condensate of the first type its anisotropy is characterized by the parameters r and $\boldsymbol{\nu}$. The same parameters determine both the anisotropy of the axially symmetric Finslerian space-time (1) and the corresponding tensor of inertial mass (28). It easy to see that at $r = 1$ the metric of axially symmetric Finslerian event space (1) degenerates into the total differential; the notion of spatial extension disappeares and in the space-time there remains the single physical characteristic, namely, time duration and it should be regarded as an interval of absolute time. Besides that at $r = 1$, in accordance with (28) there also disappeares the inertial mass of any particle.

In the case of the second, entirely anisotropic fermion-antifermion condensate its anisotropy is determined by the three parameters r_1, r_2, r_3. The same parameters determine both the anisotropy of the entirely anisotropic Finslerian space-time (7) and the corresponding tensor of inertial mass (44). With the help of the formula (7) we see that in each of the cases, namely, at

$$(r_1 = 1\,, r_2 = 1\,, r_3 = 1)\,; \qquad (r_1 = 1\,, r_2 = -1\,, r_3 = -1)\,;$$
$$(r_1 = -1\,, r_2 = 1\,, r_3 = -1)\,; \quad (r_1 = -1\,, r_2 = -1\,, r_3 = 1)\,,$$

the metric of entirely anisotropic Finslerian event space (7) degenerates into the corresponding 1-form

$$ds = dx_0 - dx_1 - dx_2 - dx_3\,; \quad ds = dx_0 - dx_1 + dx_2 + dx_3\,;$$
$$ds = dx_0 + dx_1 - dx_2 + dx_3\,; \quad ds = dx_0 + dx_1 + dx_2 - dx_3\,,$$

i.e. again into the total differential of absolute time. Besides that at all given above values of the parameters $r_1\,,\ r_2\,,\ r_3\,$, in accordance with (44) the inertial mass of any particle also disappeares. This suggests that absolute time, where the very notions of spatial extension and inertial mass become meaningless, is not a stable degenerate state of space-time; as a result of the geometric phase transition, which accompanies a spontaneous breaking of the original gauge symmetry, such an original "space-time" may turn either into the partially anisotropic space-time (1) or into the entirely anisotropic space-time (7). Thus, absolute time plays the role of a connecting link by which a principle of correspondence is satisfied for the Finslerian spaces (1) and (7).

All said above illustrates on the classiccal level an alternative mechanism of generation of mass in initially massless particles under a spontaneous breaking of the original gauge symmetry.

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