The nature of the Coulombian field: Why two formulas for the electric field created by a moving charge?

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Abstract
A derivation of the electric field created by a uniformly moving charge from its retarded position is presented. The derived formula has a general character showing how the electric field depends on the position of the point where the field is measured, on the time when the information about the creation of the field leaves the charge and on its velocity. The field described by the derived formula is mapped, showing the time variation of its components, of its magnitude and direction as measured at a given point in space and the magnitude and the direction of the field at different points in space but at the same instant of the time. Arguments are presented for the fact that a detachment of the electric field from its source takes place.

1. Introduction

The electric field created by a uniformly moving point charge can be a starting point for a relativistic approach to Maxwell's equations [1]. The scenario involves a point charge q moving with constant velocity $v = \beta c$ in the positive direction of the OX axis of an inertial reference frame S (XOY). Let S' (X'O'Y') be the rest frame of the charge. The charge is located at O', the axes of the two frames are parallel to each other and at an instant of time $t = t' = 0$ the origins O and O' are located at the same point in space.

Two formulas are known for the electric field created by moving point charge measured in the S frame. The first proposed by Heaviside [2] is

$$\vec{E} = \frac{q\vec{r}(1-\beta^2)}{4\pi\varepsilon_0 r^3(1-\beta^2\sin^2\phi)^{3/2}} \tag{1}$$

where $\vec{E}$ is the electric intensity, $\vec{r}$ is the vector from the charge to the field point where the field is required and ϕ is the angle between $\vec{r}$ and $\vec{v}$. The position of the charge is its position at the time when the field is measured at the observation point. This is called the *"present"* position of the charge.

The different ways in which electric field described by Eq. (1) can be mapped are presented by Jackson [3] and by Jefimenko [4]:

- By the density of the field lines;
- By the time variation of the components of the electric field as measured at a given point in space;
- By the time variation of the magnitude and the direction of the electric field at a given point in space.

A second formula proposed by Rosser [5] and called "electric field relative to the *retarded* position" is also known. We present a simple, transparent and relativistic derivation of that equation expressing it as a function of the instant of time t_e when the information about the creation of the field leaves the moving charge or as a function of the angle under which the information about the creation of the field is received at the observation point. We take into account the fact that the information is propagated from the charge with a speed *c* in empty space. The electric field described by that equation is mapped in order to gain support for the idea that a detachment of the Coulomb field from its source takes place [6].

2. Electric field relative to the retarded position

Let $1(x,y)=1(vt_e,0)$ be the position of the field producing charge when the information about the creation of the field leaves it at an instant of time t_e (Fig. 1). The observation point where the field is measured is $\mathrm{M}(x,y)=\mathrm{M}(r,\theta)$, its position being defined by the Cartesian coordinates $(x,\ y)$ and by the polar coordinates (r,θ). The electric field is studied in the XOY plane. The event "electric field is measured at point M when the information about the creation of the field arrives there" is characterized by a time coordinate

$$T=t_e+\frac{(r^2+v^2t_e^2+2rvt_e\cos\theta)^{1/2}}{c}\ . \tag{2}$$

When detected from the rest frame of the charge S' (X'O'Y'), the Cartesian and polar coordinates of the *same* event are [7]

$$x'=\gamma(r\cos\theta-vT) \tag{3}$$

$$y'=r\sin\theta \tag{4}$$

$$r'=(x'^2+y'^2)^{1/2}=(r^2\sin^2\theta+\gamma^2(r\cos\theta-vT)^2)^{1/2} \tag{5}$$

$$\tan\theta'=\frac{y'}{x'}=\frac{\gamma^{-1}r\sin\theta}{r\cos\theta-vT} \tag{6}$$

with $\gamma=(1-\beta^2)^{-1/2}$.

Let (E_x,E_y) and (E_x',E_y') be the components of the electric field measured from the S and S' frames respectively. They are related by [8]

$$E_x=E_x'=E'\cos\theta'=K\frac{\gamma(r\cos\theta-vT)}{(r^2\sin^2\theta+\gamma^2(r\cos\theta-vT)^2)^{3/2}} \tag{7}$$

$$E_y=\gamma E_y'=\gamma E'\sin\theta'=K\frac{\gamma r\sin\theta}{(r^2\sin^2\theta+\gamma^2(r\cos\theta-vT)^2)^{3/2}}\ . \tag{8}$$

The magnitude of the electric field is given by

$$E=(E_x^2+E_y^2)^{1/2}=K\frac{\gamma(r^2\sin^2\theta+(r\cos\theta-vT)^2)^{1/2}}{(r^2\sin^2\theta+\gamma^2(r\cos\theta-vT)^2)^{3/2}} \tag{9}$$

whereas the electric field vector is

$$\vec{E}=K\frac{\gamma((r\cos\theta-vT)\vec{i}+r\sin\theta\,\vec{j})}{(r^2\sin^2\theta+\gamma^2(r\cos\theta-vT)^2)^{3/2}} \tag{10}$$

where $K=q/4\pi\varepsilon_0$.

Equations derived above have a general character being expressed as a function of all the parameters on which the studied problem can depend.

3. Mapping the electric field relative to the retarded position

In order to simplify the problem and gain in transparence without loosing in generality, we consider that the observation point is located on the OY axis M(0,*b)* as it is shown in Fig. 2. In that case the field is measured at an instant of time

$$T_0 = t_e + \left((b/c)^2 + \beta^2 t_e^2\right)^{1/2} \tag{11}$$

the components of the electric field becoming

$$E_x = -Kc^{-2}\frac{\gamma\beta T_0}{\left((b/c)^2 + \gamma^2\beta^2 T_0^2\right)^{3/2}} \tag{12}$$

$$E_y = Kc^{-2}\frac{\gamma\, b/c}{\left((b/c)^2 + \gamma^2\beta^2 T_0^2\right)^{3/2}} \quad . \tag{13}$$

The magnitude of the electric field is given by

$$E = Kc^{-2}\frac{\gamma\left((b/c)^2 + \beta^2 T_0^2\right)^{1/2}}{\left((b/c)^2 + \gamma^2\beta^2 T_0^2\right)^{3/2}} \tag{14}$$

whereas the electric field vector is

$$\vec{E} = Kc^{-2}\frac{-\gamma\beta T_0\vec{i} + \gamma(b/c)\vec{j}}{\left((b/c)^2 + \gamma^2\beta^2 T_0^2\right)^{3/2}} \quad . \tag{15}$$

Let θ_0 be the angle made by $\vec{r}$ and $\vec{v}$ in that particular case. As we see from Fig.2, the angle θ_0 and the time t_e are related by

$$\tan\theta_0 = -\frac{b/c}{\beta t_e} \tag{16}$$

Expressed as a function of the angle θ_0, Eq.(11) becomes

$$T_0 = bc^{-1}((1+\tan^{-2}\theta_0)^{1/2} - \beta^{-1}\tan^{-1}\theta_0) \tag{17}$$

We have now all the elements to map the electric field created by the moving charge. All maps are for K=1 and c =1; the time intervals are measured in b/c unities.

We present in Fig. 3 and Fig. 4 the variation of E_x (Eq.12) and of E_y (Eq.13) respectively with the angle θ_0 under which the information which has left the charge at an instant of time t_e, is received at point M, for different values of the velocity β at which charge moves in the positive direction of the OX axis but for a constant value of bc^{-1}.The relationship between t_e and θ_0 results from Fig. 5.

Eq. (14) enables us to construct a "polar diagram" or an "electric field contour" presented in Fig. 6. The length of the position vector on such a diagram equates the strength of the electric field E_{θ_0} i.e. at different instants of time t_e. The origin of the diagram coincides with the observation point M.

Eq. (9) enables us to take a "snapshot" of the electric field i.e. to represent the electric field at different points in space but at the same instant of time. Let t_e=0 be the instant of time when the information about the creation of the field starts to propagate. Under such conditions at all the points of a circle of radius r the field is measured at an instant of time

$$T = rc^{-1} \tag{18}$$

whereas Eqs.(7), (8), (9) and (10) become

$$E_x = K\frac{\gamma^{-2}(\cos\theta-\beta)}{r^2(1-\beta\cos\theta)^3} \tag{19}$$

$$E_y = K\frac{\gamma^{-2}\sin\theta}{r^2(1-\beta\cos\theta)^3} \tag{20}$$

$$E = K\frac{\gamma^{-2}(1-2\beta\cos\theta+\beta^2)^{1/2}}{r^2(1-\beta\cos\theta)^3} \tag{21}$$

and

$$\vec{E} = K\frac{\gamma^{2}(\vec{r}-\vec{\beta}r)}{r^3(1-\beta\cos\theta)^3} \tag{22}$$

We recognize in Eq.(22) the equation derived by Rosser, knowing now the conditions under which it holds. Our fully relativistic derivation of it reveals the physical meaning of the physical quantities, which intervene.

With Eq.(21) as basis, we construct the polar diagram presented in Fig. 7. The length of the position vector equates the magnitude of the electric field E measured along a direction which makes an angle with the positive direction of the OX axis, measured at a point $\mathrm{M}(r,\theta)$ located on a circle of radius *r* at an instant of time *r/c*, the information about the creation of the field having left the charge at an instant of time t_e=0. In order to obtain a full image of the situation, we present in Fig. 8 a circle of radius *r* for which Fig. 7 was constructed. Translating the vector E with its origin to point $M(r,\theta)$ we obtain a full representation of the "snapshot".

We stress now some peculiarities of the retarded position approach:

-The space-time coordinates of the event "electric field is measured at a given point in space at a given instant of time" are created by a signal that propagates with velocity *c* and we consider that they are detected using the radar detection procedure [9].

-The approach makes a net distinction between the situation when the point charge is "outgoing" $(\theta_0=0)$ or it is "incoming" $(\theta_0=\pi)$. In the first case we have (Eqs.(19), (20))

$$E_x = Kr^{-2}\frac{1+\beta}{1-\beta} = Kr^{-2}D \tag{23}$$

$$E_y = 0 \tag{24}$$

whereas in the second case the same equations lead to

$$E_x = Kr^{-2}\frac{1-\beta}{1+\beta} = Kr^{-2}D^{-1} \tag{25}$$

$$E_y = 0 \tag{26}$$

The electric field created by the moving point charge has much in common with a wave. Among others, as in the case of a wave, we can study it at the same point in space but at different instants of time or at a given instant of time but at different point in space. The presence of the Doppler factor D in some results is a plea for that idea. Comparing Eq.(1) derived relative to the present position and Eq. (21) derived relative to the retarded position, we see that the essential difference between them consists in the fact that in the first case the angular parameter intervenes at the second power $(\sin^2\theta)$ whereas in the second it intervenes at the first power $(\cos\theta)$. That explains why

Heavisides's equation (1) makes no distinction between "incoming charge" and "outgoing charge", whereas Rosser's formula (Eq.(21)) makes that distinction.

4. Angular velocity of the electric field vector at observation point M(0,*b*)

Observation of the electric vector at a given point in the XOY plane suggests that there a continuous rotation of it takes place. We consider the situation at point M(0,*b*), calculating the angular velocity at which it takes place. At an instant of time T, the electric vector makes an angle α with the positive direction of the OX axis given by

$$\tan\alpha = -E_y / E_x = \frac{bc^{-1}}{\beta T_0} \tag{27}$$

Differentiating both sides of Eq. (27) and rearranging the terms, we obtain for the angular velocity

$$\omega = \frac{d\alpha}{dT_0} = \frac{\beta bc^{-1}}{\left(bc^{-1}\right)^2 + \beta^2\left(t_e + \left(bc^{-1}\right)^2 + \beta^2 t_e^2\right)^{1/2}} \tag{28}$$

and we present its variation with t_e for different values of β in Fig. 9.

5. Field lines in the Coulombian field

We consider the problem of the field lines in the Coulombian field. In order to keep the problem as simple as possible, we consider the situation at an instant of time t_e=0, when the charge is instantly located at the origin O of the S frame. Let $\mathrm{M}(x,y) = \mathrm{M}(r,\theta)$ be the observation point. Combining Eqs. (19) and (20) we obtain for the slope of the field line at that point

$$\frac{dy}{dx} = \frac{\sin\theta}{\cos\theta - \beta} \quad . \tag{29}$$

Expressing the left hand side of Eq. (29) as a function of polar coordinates ($dy = \sin\theta\, dr + r\cos\theta\, d\theta$; $dx = \cos\theta\, dr - r\sin\theta\, d\theta$) and separating the variables we obtain the differential equation

$$\frac{dr}{r} = \frac{1-\beta\cos\theta}{\beta\sin\theta} d\theta \tag{30}$$

which integrated leads to ($\int \frac{d\theta}{\sin\theta} = \ln\tan\frac{\theta}{2} + C_1$; $\int \frac{\cos\theta}{\sin\theta} d\theta = \ln\sin\theta + C_2$)

$$\frac{r\sin\theta}{\left(\tan\frac{\theta}{2}\right)^{1/\beta}} = C \quad . \tag{31}$$

We represent the field lines in Fig. 10. As we see, a continuous curvature of the field lines takes place, which becomes more and more manifest as the distance between point charge and observation point increases. Such a curvature of the field lines takes place in the case of an accelerating point charge as well [6], [10], the field being considered from the retarded position of the charge. Our results are an argument for the fact that acceleration is not the single source for the curvature of the field lines. Peculiarities of the field lines are presented in Fig. 10 as well.

6. Magnetic field relative to the retarded position

The point charge being at rest in the S' frame, no magnetic field is detected here ($B' = B'_x = B'_y = 0$). What is detected is the electric field $E'(E'_x, E'_y)$. From the S frame we detect a moving electric field and it is customary to say that a magnetic field is present, which has in that case only an OZ component B_z obtained from the transformation equation

$$B_z = \gamma(B'_z + vc^{-2}E'_y) = \gamma K\beta c^{-1}\frac{r\sin\theta}{\left(r^2\sin^2\theta + \gamma^2(r\cos\theta - vT)^2\right)^{3/2}} \tag{32}$$

or in vector form

$$\vec{B} = \gamma K c^{-1}\frac{\vec{\beta} x\vec{r}}{\left(r^2\sin^2\theta + \gamma^2(r\cos\theta - vT)^2\right)^{3/2}} \tag{33}$$

In the particular case when $t_e = 0$ and $T = r/c$, Eq.(33) becomes

$$\vec{B}_{t_e=0} = \gamma^{-2}Kc^{-1}\frac{\vec{\beta} x\vec{r}}{r^3(1-\beta\cos\theta)^3} \tag{34}$$

in accordance with Rosser [1] and Jefimenko [4].

In order to map the magnetic field we consider the situation at point M(0,*b*), where Eq.(32) becomes

$$B_z = \gamma c^{-3}\beta K\frac{bc^{-1}}{\left((bc^{-2}) + \gamma^2\beta^2T_0^2\right)^{3/2}} \quad . \tag{35}$$

As we see, a time variation of B_z takes place, the magnetic field having only an OZ component. The variation of B_z with the angle θ_0 under which information about the creation of the field is received at point M is presented in Fig. 11.

7. Arguments for the detachment of the Coulombian field from the source that induces it

Harpaz [6] considers that the detachment of the Coulombian field from the charge that induced it is a fundamental concept in explaining the radiation by an accelerating charge. The invoked argument is the fact that the field at a given point is determined from the retarded position of the charge. Our derivation shows that this argument holds in the case of the Coulombian field created by a uniformly moving charge as well. We present a convincing example for the fact that the *Coulombian field* is an independent physical entity that should be considered on the same ground as matter particles.

Consider a point charge *q* located at the origin O' of its rest frame S'(X'O'Y') as it is shown in Fig. 11. An earthed metallic sphere shields the external space from the charge inside it. The electric potential of the charge is maintained constant by the source E. Consider that at an instant of time $t'_e = 0$ the contact with Earth is cut for an interval of time Δt. After a time *t* of propagation the electric field is confined between two spheres of radius *ct* and $c(t+\Delta t)$ respectively, where the emitted energy is confined and which has lost any contact with the source *q* and we are obliged to consider it as an independent physical entity. It is an instructive exercise to consider the same problem from the reference frame S, the problem having much in common with the blackbody radiation field, which radiates more energy in the direction of the motion then in the other directions along which the radiation is detected [11].

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Captions

Fig. 1. Scenario for deriving the electric field at point M (r,θ) from the retarded position 1 of the moving charge q.

Fig. 2. Scenario for deriving the electric field at point M (0,b) from the retarded position 1 of the moving point charge.

Fig. 3. Variation of the OX component of the electric field (E_x) with the angle θ_0 under which the information about the creation of the field is received at the observation point M (0,b), for different values of the velocity $v = \beta c$ of the charge, moving in the positive direction of the OX axis, illustrating Eq. (12). The curves correspond to: a) β=0.1; b) β=0.3; c) β =0.6; d) β=0.9.

Fig. 4. Variation of the OY component of the electric field (E_y) with the angle θ_0 under which the information about the creation of the field is received at the observation point M (0,b), for different values of the velocity $v = \beta c$ of the charge, moving in the positive direction of the OX axis, illustrating Eq. (13). The curves correspond to: a) β=0.1; b) β=0.3; c) β =0.6; d) β=0.9.

Fig. 5. Relationship between the time t_e at which the information about the creation of the field is emitted and the angle θ_0 under which it is received at the observation point M (0,*b*), illustrating Eq. (16), for β=0.9.

Fig. 6. Magnitude of the electric field measured at the point M (0,*b*) for different values of the angle θ_0 under which the information about the creation of the field is successively received. Polar coordinates are used for illustrating Eq. (14). The length of the "position vector" equates the magnitude of the electric field and makes an angle θ_0 with the velocity vector $\vec{v}$. The curves correspond to: a) β=0.1; b) β =0.6; c) β=0.9.

Fig. 7. The magnitude of the electric field measured at different points of a circle with radius *r* at the same instant of time *r/c* (snapshot) when the information about the creation of the field arrives at that points. Polar coordinates are used and the length of the "position vector" equates the magnitude of the electric field measured at a point M(r,θ).

Fig. 8. A "snapshot" of the electric fields measured at different points of a circle of radius *r* at the same instant of time *r/c*. The corresponding magnitudes of the measured electric fields are obtained from Fig. 7.

Fig. 9. The angular velocity ω at which the electric field $\vec{E}$ rotates around the observation point M (0,*b*), as a function of the time t_e when the information about the creation of the field leaves the charge.

Fig. 10.a. The shape of the field lines, characterizing the electric field created by a moving point charge for $\beta = vc^{-1} = 0.3$ and different values of the constant C introduced by the integration of Eq. (30). The information about the creation of the field leaves the charge at an instant of time t_e=0, when it is instantly located at the origin of the S reference frame. The constant C is defined by imposing the condition that the point M (r_0, θ_0) is located on the field line. For the field lines presented, r_0 is maintained constant, θ_0 taking the values inscribed on the corresponding field lines.

Fig. 10.b. The shape of the field lines characterized by the same value of the constant C introduced by the integration of Eq. (30) but for different values of the velocity $v = \beta c$ of the charge. The situation corresponds to the case when the information about the creation of the field is emitted at an instant of time t_e=0, when the charge is instantly located at the origin O of the S (XOY) reference frame. The curves correspond to: a) β=0.1; b) β=0.3; c) β =0.6; d) β=0.9.

Fig. 11. The variation of the magnitude of the magnetic field B_z measured at the observation point M (0,*b*) with the angle θ_0 under which the information about the creation of the field is received at that point. The magnetic field does not change its direction at that point. The curves correspond to: a) β=0.1; b) β=0.3; c) β =0.6; d) β=0.9.

Fig. 12. Experiment in order to prove the detachment of the Coulombian field from the stationary point charge *q*.

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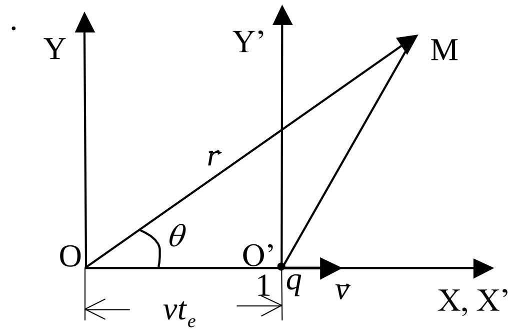

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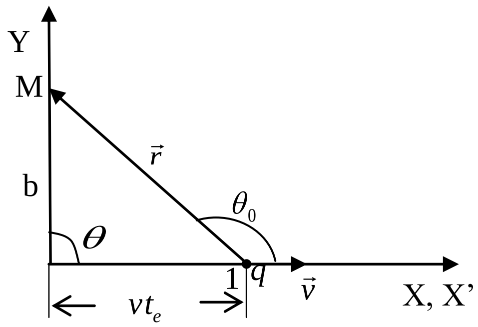

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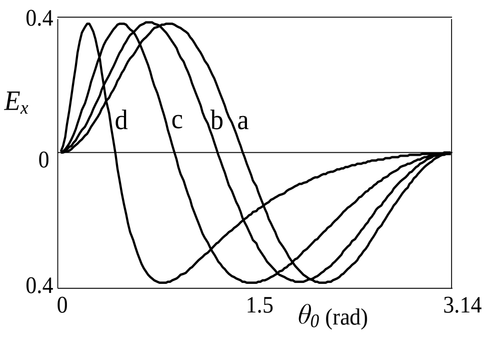

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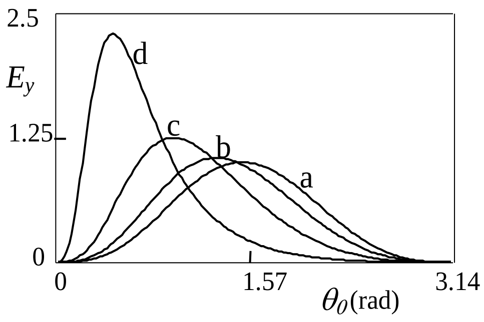

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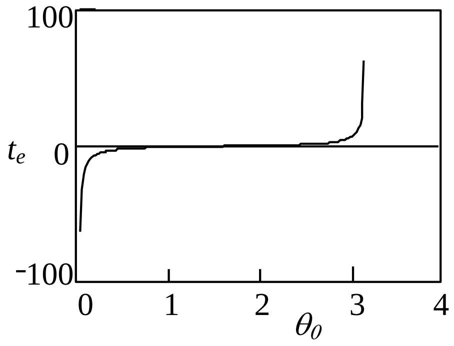

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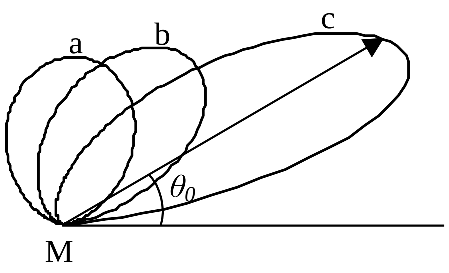

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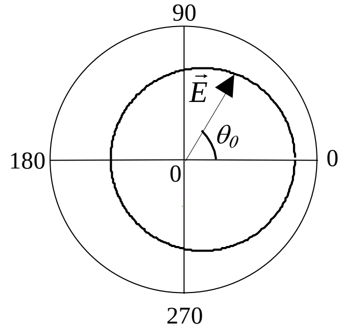

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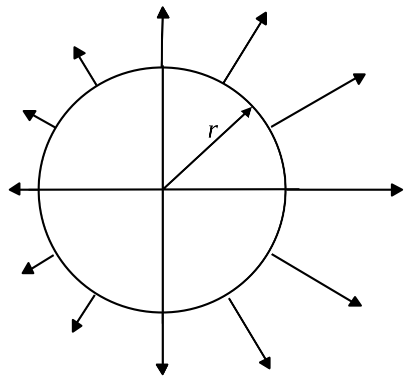

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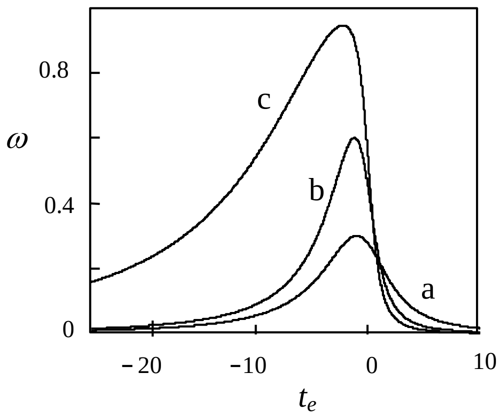

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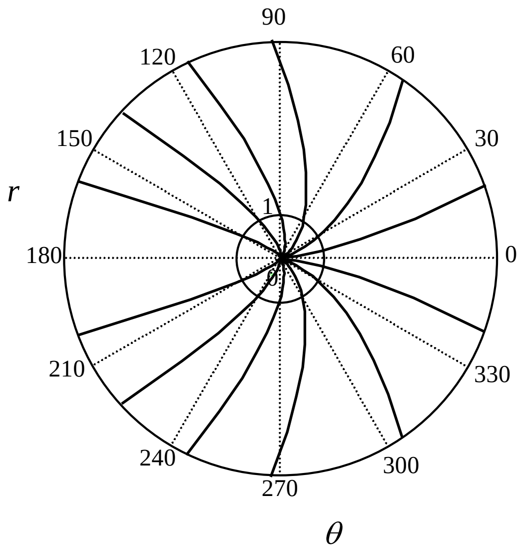
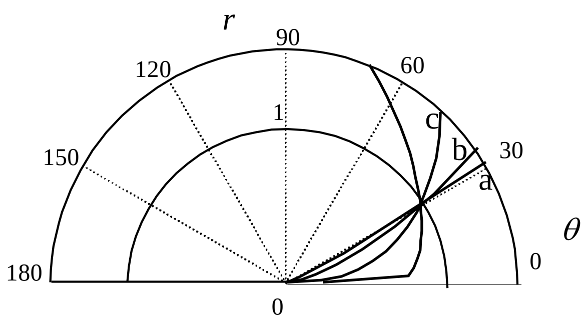


Fig. 10.a.



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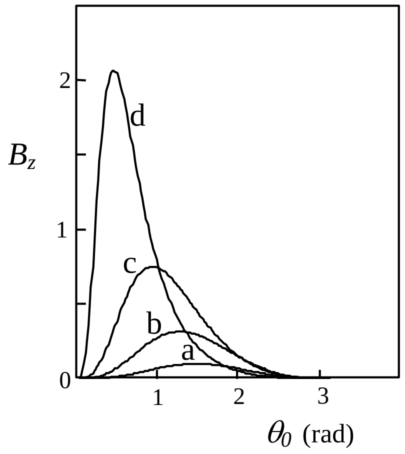

Fig. 11

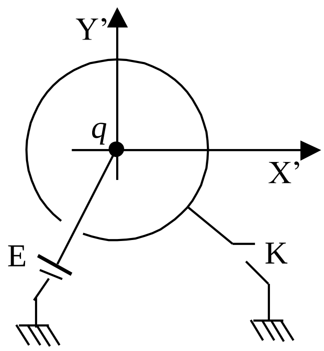

Fig. 12.