From sightings to special relativity: A route untrodden

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Irrespective of their states of motion relative to one another, observers momentarily clustering together at a single spot in empty space always ***sight*** the same global set of material coincidences around at the instant they are together. "Observers" can here be cameras / instruments, and "sight" can mean "detect from incoming electromagnetic radiations". This result serves as an attractive point of departure for learning the basic kinematical features of special relativity. We name it the "camera principle" for ease of reference.

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1. Introduction : The clock principle

Suppose some event E happens at a point $\mathbf{r} \equiv (x, y, z)$ in space, and at an instant t in time. The space-time coordinates x, y, z and t appearing here are supposed to have been measured in a certain frame of reference S. The observer O who watches E is at the origin (0,0,0) of S. In this situation, O "sights" the event at a later time

$$T = t + (r / c) , \qquad [r = (x^2 + y^2 + z^2)^{1/2}], \qquad (1)$$

c being the speed of the *signals* that bring the news of E over to O. In special relativity, we are concerned mostly with light or other electromagnetic signals commonly generated by events [1]. The velocity c of these signals in empty regions of space is found to be independent of the direction of their propagation when measured relative to certain structures of reference known as *inertial frames*. We assume S to be one such inertial reference frame. Experimental studies have shown that any other frame S' will be inertial if and only if it is in a state of *uniform* motion relative to S.

Lorentz transformations have usually been the principal gateway to special relativity in elementary physics courses. The time coordinates appearing in these transformations are, in the notation of Eq.(1), exclusively the t's always. Never are they the T's -- which, of course, are what O's "wrist watch" invariably displays. This raises a minor problem, in that it seems necessary to place a properly synchronized *different* clock at the site $\mathbf{r} \equiv (x, y, z)$, too, to be able to measure the remote time t. Einstein's original formulation of special relativity in this way deploys manifold infinities of clocks, one at each point of space in every conceivable inertial reference frame, and gives directions as to how the clocks within each frame are to be pre-synchronized as well [2].

Yet there is a more facile route to determining t. With the arrival time T of E's signals read off a single clock placed at (0,0,0), the observer O sitting there can always *compute* the time t of the event's occurrence straightaway from Eq.(1) **!** The results of this procedure, which amounts to *defining* t via Eq.(1), will obviously be identical to whatever is obtainable from Einstein's multiple clocks.

In its new-found rôle as a substitute for multiple clocks, Eq.(1) merits a name for itself. We opt for calling it the "clock principle". Implicit in this principle is the assurance of c's isotropy**;** this means that the velocity of electromagnetic signals in empty space as measured relative to a given inertial frame S will have an unchanging magnitude, regardless of the *direction* in which the signals are propagating [3].

2. The camera principle and other inputs

Besides the "clock principle", the current paper relies on yet another input in a pivotal rôle. Cryptically designated the "camera principle", the latter expresses the fact that localized observers when instantaneously close to one another always end up *sighting* one and the same set of physical coincidences around, no matter how different their individual states of motion are. We are referring here to the coincidences of all *materially identifiable* locations that can be "seen" by the different observers in the global spaces of each other. What lends particular plausibility to this principle is a theory much older than relativity **--** the wave theory of light. Observers in a cluster are pictured in this theory as receiving the same group of carrier waves, all of these with the same critical modulations stamped in. Thus, though we shall be concerned in the sequel only with observers resting in inertial frames, our "camera principle" should remain true for non-inertial observers as well.

A third input we shall use is the principle of "equidistance patterns". Unfortunately, this principle has an excessive contextual encumbrance that makes its description in general terms unviable. We defer its explanation until the time the need for it explicitly arises. See Sec.7.

The last of the inputs we look to is the well-known principle of interchangeability of inertial frames. This tells us that it is always permissible to swap the rôles of any two inertial frames S and S' in all valid relationships involving measurements carried out from them.

3. About this paper and its *raison d'être*

It is true that not much headway is possible in special relativity without good proficiency in the use of Lorentz transformations. Yet it is equally true that these transformations do not by any means provide a readily intelligible *introduction* to the subject. Their stuff is esoteric despite outward appearances, and can only elicit perplexities if dinned into the ears of the uninitiated. Alive to this pedagogic problem, authors and educators in the past few decades have been on a search for alternative (or perhaps complementary) avenues to approach relativity's beginnings. Their works [4] have doubtless registered appreciable successes, but, by being concerned mostly with second-level effects like aberration and Doppler shifts, they appear to be missing out on some of the kinematic core issues involved.

The core issues left unaddressed include length contractions and time dilations. They also include the indeterminate time orderings of spacelike separated events and the related collapse of absolute simultaneity. As is well known, it is precisely these latter aspects of relativity that cause the most confusion in the minds of first-time learners. Against this backdrop, the current presentation specifically aims at promoting a better understanding of these exotic features of special relativity, too, with the aid of a couple of simple inputs -- chiefly the "camera principle" and the "clock principle". It assumes prior knowledge of neither Lorentz transformations nor the jargons associated with them -- like "observation" $\neq$ "looking at", and such other obliquities. An illustrative example discussed in the next two sections will clinch the essential logic of our approach. The arguments put together in these sections should clearly be accessible to any intelligent *layman*. This preliminary example will be followed by a more formal treatment of some standard results and their ramifications in the later sections.

4. Illustrative example

Imagine an ultrafast railroad train of immense length coasting ahead in uniform motion towards a well-illuminated straight tunnel. This tunnel measures several light-minutes through from one end to the other [5] . Only plain vacuum pervades its interior and outside.

The train's speed is stupendous. It fails to match the velocity of light by only a small percentage. There are six passengers P0 through P5 aboard the train. They are seated at precisely equidistant points -- Ms P0 at the train's tail end, Ms P5 at its forward extremity, and the rest in between. Likewise, six watchmen W0 through W5 are stationed inside the tunnel, and these, too, are in precisely equidistant positions **--** Mr W0 at the entrance, Mr W5 at the exit, and the rest in between.

Our penultimate watchman W4 has his eyes directly over his *ears*. At some point of time as the train is hurtling through his tunnel (see Figure 1), he catches sight of both

(***i***) P0 and W0 waving hands at each other at the tunnel's entrance with his *left* eye -- the event "[P0**:**W0]" **;** and

(***ii***) P5 and W5 similarly greeting each other at the tunnel's exit with his *right* eye -- the event "[P5**:**W5]".

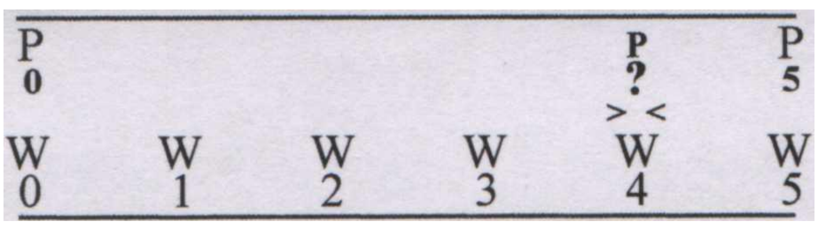

Figure 1. Momentary view picked up by [W4] 's eyes

Still, W4 does *not* conclude from this scenario that the train and the tunnel have equal lengths. To see why, suppose, for example, that the tunnel's length is 5 light-minutes. Then W4 is 4 light-minutes away from W0, and our "clock principle" [Sec.1] tells us that the light signals from the event [P0**:**W0] must have arrived at W4's location only after spending four long minutes in transit. In contrast, the signals from [P5**:**W5] would by the same principle have reached that spot in a single minute, because W4 is just 1 light-minute away from W5. By W4's reckoning, therefore, the coincidence [P0**:**W0] is an event long past -- one that happened far, *far* ***earlier*** than [P5**:**W5]. Hence, at the exact moment when P0 met W0 at the tunnel's entrance, P5 sitting at the train's other end must have been way behind the exit watchman W5, and was thus *far inside* the tunnel**!** This reasoning makes the train ***considerably*** shorter than the tunnel.

5. Illustrative example : continued

At the precise time when Mr W4 concurrently sights the events [P0**:**W0] and [P5**:**W5], it is conceivable that some *passenger* Ms P? would be darting past W4 (Figure 1) and greeting him as she did so. This P? would *not* be P4, however. To see why, fancy for a moment that our train can be exactly as fast as light itself. Then passenger P0 riding it stays permanently *abreast* of the rays of light emanating from [P0**:**W0] and racing towards W4. This can only result in P0 *herself* appearing before W4 when he sights [P0**:**W0] [6] **!** The train, though, is a shade slower than light actually. It is therefore more realistic to reckon **P1 --** rather than P0 **--** to be against W4 at the instant in question. Does the idea sound crazy? It shouldn't. All it means is that the light signals from [P0**:**W0] have been on a hot chase behind P1, and have at last managed to close in on her only at W4's site. The pursuit proving to be so long-drawn out needn't occasion surprise, either. It only reflects the fact that whatever edge even light has over our ultrafast train is exceedingly narrow.

So we may, *with the train's speed right,* legitimately set P? ≡ P1 in Figure 1. This brings W4 and P1 at one spot together when the light signals from [P0**:**W0] and [P5**:**W5] concurrently arrive there. Accordingly, by our "camera principle" (Sec.2), P1 is obliged to sight *exactly* the same set of identifiable coincidences around as W4 does. In particular, we conclude that P1 *also* sights the two coincidences [P0**:**W0] and [P5**:**W5] simultaneously.

But, as a true denizen of the train rather than the tunnel, passenger P1 sees these coincidences in a manner different from how watchman W4 sees them. The view she picks up is shown in Figure 2**:**

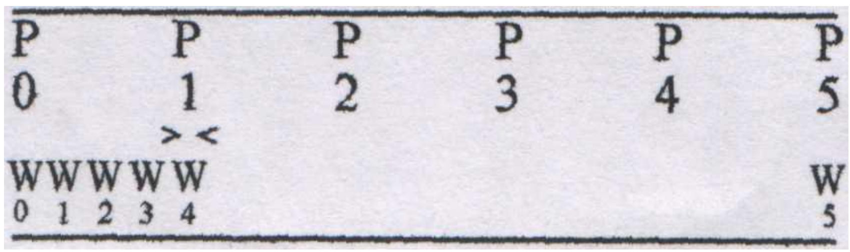

Figure 2. Momentary view picked up by P1 's eyes

It is a well established fact in physics that when a tunnel as a frame of reference is inertial, any train coasting through it in uniform motion will also be inertial. Accordingly, prior experiments would have convinced P1 that light travels at the same speed in all directions relative to her train as well. P1 can therefore appeal to our "clock principle" (Sec.1) with no misgivings bothering her now, and she draws the conclusion that the *distant* event [P5**:**W5] of Figure 2 must have occurred at a time far earlier in the past than the nearby [P0**:**W0] [7] **!** So when [P5**:**W5] was happening long long ago, P0 could only have been *outside* the tunnel -- far, far behind W0**!** This means that P5 was already *exiting* the tunnel when P0 had not entered it at all yet. P1 readily convinces herself by this reasoning that it is the *tunnel* that is pretty much shorter than her train.

The conclusions of W4 and P1 are thus, by classical reckoning, totally *conflicting* over both

(***a***) the time orderings of the events [P0**:**W0] and [P5**:**W5] [8] **;**

and

(***b***) the relative lengths of the tunnel and the train.

Notice that the (***b***)-inferences are in fact logical implications the (***a***)-ones. Pre-relativistic physics would have dismissed this tangle as a frivolous muddle resulting from an *invalid* application of the clock principle to Figure 2 **--** invalid in its view since the velocity of light was supposed to be unisotropic relative to the moving train in earlier days. Special relativity, in contrast, sees no muddle at all here. It upholds both W4's and P1's conclusions as equally valid by declaring them to be *relative* **--** one to the tunnel and the other to the train.

6. The poetic visions of kinematics

The revelations of our introductory example cry for a well-knit generalization with plural dimensions added in. Consider, in lieu of the train and the tunnel, a pair of inertial frames of reference S and S'. A *series* of coincidences $[P_\rho' : P_\rho]$, ($\rho = 0,1,2, \ldots$), are concurrently *sighted* at some common location $[O' : O]$ by ***both*** an S-observer O and an S'-observer O' [cf. the camera principle (Sec.2) **;** see Figures 3a and 3b **;** to avoid cluttering up, only a single P_ρ and a single P_ρ' have been shown in these diagrams]. Further, the P_ρ's in the frame S are all supposed to lie in a *straight line* ξ parallel (or anti-parallel) to the direction of S''s motion relative to S. In this circumstance, ξ will be the common trajectory in S of all the "moving" S'-sites P_ρ' (Figure 3a). As a consequence, it turns out likewise that all the "moving" S-sites P_ρ, too, have a common straight line trajectory ξ' in S' on which the P_ρ''s lie (Figure 3b).

Our "clock principle" [Eq.(1), Sec.1] tells the observers O and O' that whatever coincidences $[P_\rho' : P_\rho]$ they are sighting at the current moment are actually past events. Let $\beta > 0$ be the ratio of S''s speed to the *direction-independent* speed of light as measured in the frame S, and $\beta' > 0$ likewise the ratio of S's speed to the *direction-independent* speed of light as measured in the frame S'. [It will be proved later that $\beta = \beta'$, see Eqs.(11-12), Sec.8.] Then O draws the inference that during the time the light signals transit across the distance $r_\rho = O P_\rho$, the site $P_\rho{}'$ attached to the "moving" frame S' must have shifted to a different location $P_\rho{}'^{\,!}$ as shown in Figure 3c -- with $P_\rho P_\rho{}'^{\,!} = \beta r_\rho$. The other

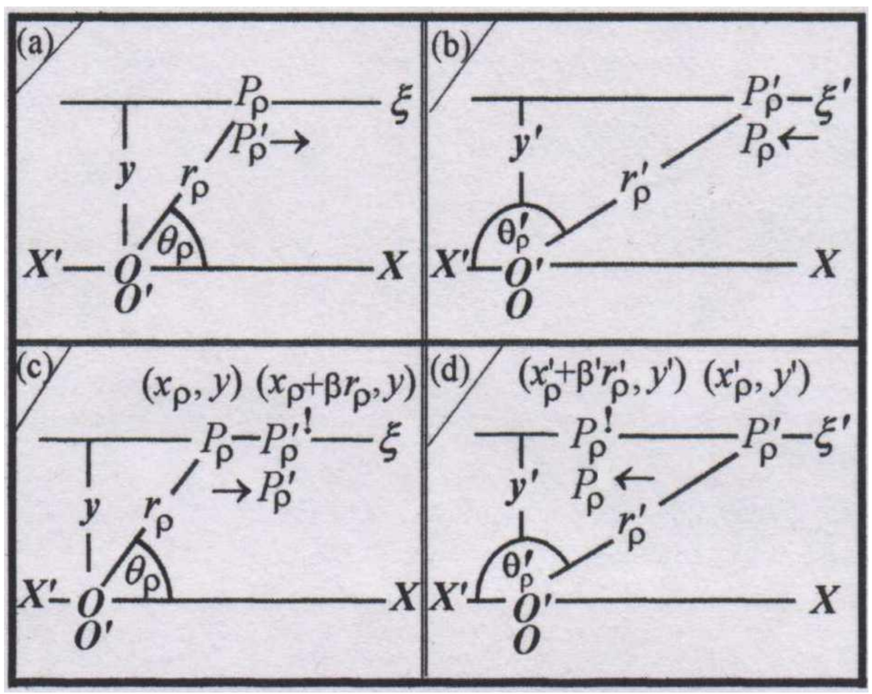

Figure 3. (a) The coincidence $[P_\rho{}' : P_\rho]$ as seen by O ; (b) The same event as seen by O' ; (c) O 's mind's eye picturing the moving S'-site $P_\rho{}'$ as having shifted to $P_\rho{}'^{\,!}$; and (d) O' 's mind's eye picturing P_ρ as having moved to $P_\rho{}^{!}$. Note that O and O' measure their abscissas x and x' in opposite directions. O 's version of electrodynamics in S always attributes to $P_\rho{}'$ the *unsighted* $P_\rho{}'^{\,!}$'s coordinates. A similar remark applies to O' 's version of it in S' as well.

observer O' similarly updates the position of the "moving site" P_ρ to $P_\rho{}^{!}$

with $P_\rho' P_\rho^{!} = \beta' r_\rho'$ (Figure 3d)[9]. The orthodox relativity jargon proclaims in this context that *O* "sees" P_ρ' at P_ρ, but "observes" it at $P_\rho'^{!}$ -- and likewise that O' "sees" P_ρ at P_ρ', but "observes" it at $P_\rho^{!}$. We opt, however, for a more poetic yet less puzzling language **:** *O* and O' respectively see P_ρ' at P_ρ and P_ρ at P_ρ' with their *actual* eyes, and P_ρ' at $P_\rho'^{!}$ and P_ρ at $P_\rho^{!}$ with their "*mind's* eyes".

7. Mathematical analysis

Imagine, *for a moment,* the sites P_ρ' to be **equidistant** in their native frame S' (Figure 3b). Then the sites P_ρ in the other frame *S* (Figure 3a) at which these are sighted by *O* would ***not*** be equidistant. This is because *O*'s *actual* eyes (or instruments) would have detected the P_ρ''s in their motion only at the "retarded" positions they happened to be in at *varying* past times. *O*'s mind's eye, by contrast, is supposed to suffer no such limitation. It looks reasonable therefore to expect the corresponding $P_\rho'^{!}$'s (of Figure 3c) as *computed* by this mind's eye to take up neatly equidistant positions in *S* whenever the original P_ρ''s ***are*** equidistant in S'.

This, in fact, constitutes the principle of "equidistance patterns" we mentioned without description in Sec.2 [10]. It leads to the inference

$$P_0'P_\rho' / P_0'P_\nu' \;=\; P_0'{}^{!}P_\rho'{}^{!} / P_0'{}^{!}P_\nu'{}^{!} = \; \rho / \nu \qquad (2)$$

for any relevant ρ and ν. In this context, we do ***not*** assume a *common* system of measuring units to have been established in S and S' at the very outset. This is a perfectly legitimate strategic option, and we are exercising it on purpose [11]. It necessitates treating the lengths $P_0'{}^{!}P_\rho'{}^{!}$ and $P_0'{}^{!}P_\nu'{}^{!}$ in the frame S, and $P_0'P_\rho'$ and $P_0'P_\nu'$ in the frame S', as having been measured in arbitrarily chosen *different* units. Accordingly, when Eq.(2) is re-written in the form

$$P_0'P_\rho' / P_0'{}^{!}P_\rho'{}^{!} \;=\; P_0'P_\nu' \;/ P_0'{}^{!}P_\nu'{}^{!} \;=\; \gamma > 0, \qquad (3)$$

the constant γ, far from being automatically fixed once for all, depends critically on the choice of measuring units in S and in S', and necessarily *changes* its value whenever either unit is altered for any reason.

In terms of coordinates depicted in Figures 3c and 3d, Eq. (3) assumes the form [12]

$$\frac{-(x_\rho' - x_0')}{(x_\rho + \beta\, r_\rho) - (x_0 + \beta\, r_0)} = \frac{-(x_\nu' - x_0')}{(x_\nu + \beta\, r_\nu) - (x_0 + \beta\, r_0)} = \gamma\,. \tag{4}$$

If we hold ν fixed, and treat only ρ as variable, Eq.(4) acquires a simpler look

$$x_\rho' = -\gamma\,(x_\rho + \beta r_\rho) + \alpha, \tag{5}$$

where α is yet another constant. As long as one and the same fixed pair of events are chosen as $[P_0' : P_0]$ and $[P_\nu' : P_\nu]$, Eq. (5) holds with the same γ and the same α for all ρ, whatever be the number ν of equal divisions appearing in $P_0'P_\nu'$. But the P_ρ''s can be densely laid out everywhere along the line ξ' by just making ν indefinitely large. In this circumstance, reasons of geometric *continuity* lead us to the conclusion that the relationship obtained by dropping the subscripts ρ from Eq.(5), namely

$$x' = -\gamma\,(x + \beta r) + \alpha, \qquad [r = (x^2 + y^2)^{1/2}], \tag{6}$$

must hold good for ***any*** $P' \equiv (x', y')$ sighted in coincidence with $P \equiv (x, y)$ concurrently with the $[P_\rho' : P_\rho]$'s at the location $[O' : O]$.

8. Mathematical analysis : continued

Now we take the last crucial step by invoking the principle of interchangeability of inertial frames (Sec.2). This principle affirms that a twin relationship

$$x = -\gamma'(x' + \beta' r') + \alpha', \quad [r' = (x'^2 + y'^2)^{1/2}], \tag{7}$$

must also hold good in precisely the same context as above, with

$$\gamma' = P_0 P_\rho / P_0{}^! P_\rho{}^!. \tag{8}$$

When rationalized, the two equations (6) and (7) take on the forms

$$\begin{aligned} &\gamma^2(1-\beta^2)\,x^2 + 2\gamma x x' + x'^2 \\ &\qquad - 2\gamma\alpha x - 2\alpha x' + (\alpha^2 - \gamma^2\beta^2 y^2) = 0\,; \end{aligned} \tag{9}$$

and

$$\begin{aligned} &x^2 + 2\gamma' x x' + \gamma'^2(1-\beta'^2)\,x'^2 \\ &\qquad - 2\alpha' x - 2\gamma'\alpha' x' + (\alpha'^2 - \gamma'^2\beta'^2 y'^2) = 0. \end{aligned} \tag{10}$$

The obvious implication is that the corresponding coefficients of Eqs.(9) and (10) must be *proportional*. This condition leads to

$$\gamma\gamma' = 1/(1-\beta^2) = 1/(1-\beta'^2), \tag{11}$$

$$\beta = \beta' \;\; ; \quad \alpha = \alpha' = 0\,; \tag{12}$$

$$\gamma'^{1/2} y' = \gamma^{1/2} y. \tag{13}$$

When updated using Eq.(12), Eqs.(6) and (7) of the preceding section read

$$x' = -\gamma\,(x + \beta\, r), \quad x = -\gamma'\,(x' + \beta\, r'). \tag{14}$$

These relations raise the question of their own consistency with Eq.(13), since, after all, y and y' are readily expressible in terms of x, r and x', r'.

Thanks to Eq.(11), Eqs.(14) have the immediate corollaries

$$r' = \gamma\,(r + \beta\, x)\,, \quad r = \gamma'\,(r' + \beta\, x')\;; \tag{15}$$

$$x'\,(r + \beta\, x) = -\, r'\,(x + \beta\, r). \tag{16}$$

It is a simple exercise now to recover Eq.(13) in the form $\gamma'\,(r'^{\,2} - x'^{\,2})$ $= \gamma\,(r^2 - x^2)$ from Eqs.(11) and (16). This not only vouchsafes the mutual consistency of all our equations, but in a way also evidences the innate truth of special relativity's logic *per se* **:** Idle fancies would have been prone to internal contradictions.

9. Relativistic length contractions

We have so far been expressing lengths in the frames S and S' in arbitrarily chosen different units. Now that this strategy has paid off, there is no need to stick to it any more. Accordingly, we do assume henceforth a *common* system of units for all conceivable inertial reference frames. The most natural way to establish the desired common standards consists in ***readjusting*** the units of length in S and S' so as to equalize the numerical values of y and y' measured in these frames. Thanks to Eq.(13), (Sec.8), this simple maneuver ensures universal compliance of the equality $\gamma = \gamma'$ in the sequel. The key consequences flowing in this circumstance from several of our earlier relations [(3), (8) and (11-15)] are

$$y = y' \; ; \; \beta = \beta' \; ; \tag{17}$$

$$P_0' P_\rho' / P_0'^{!} P_\rho'^{!} = P_0 P_\rho / P_0^{!} P_\rho^{!} = (1-\beta^2)^{-1/2} \tag{18}$$

$$= \gamma = \gamma' > 1 \; ; \tag{19}$$

$$x' = -\gamma(x + \beta r), \quad x = -\gamma(x' + \beta r') \; ; \tag{20}$$

$$r' = \gamma(r + \beta x), \quad r = \gamma(r' + \beta x'). \tag{21}$$

A universal *time* standard, too, can be established next by stipulating that the omnidirectional speed of light in empty space shall have one and the same numerical value c in all inertial frames of reference.

Eq.(18-19) expresses the so-called *length contraction effect* of special relativity. Here the observer O' residing in the frame S' compares a length $P_0 P_\rho = \lambda$ that is at rest in the *other* frame S (Figure 3a) ***not*** with $P_0' P_\rho' = \lambda'$ (Figure 3b), but rather with $P_0^{\,!} P_\rho^{\,!} = \lambda^{!}$ (Figure 3d). While $P_0' P_\rho' = \lambda'$ is how O' instantaneously ***sights*** $P_0 P_\rho = \lambda$ from her native frame S' with her actual eyes, $P_0^{\,!} P_\rho^{\,!} = \lambda^{!}$ is what her mind's eye pictures to be the changed length of λ in its *updated* position. The update results from the "moving" host frame S having in the mean time *carried* λ leftward relatively to S'. And it is the *mind*'*s eye*'s construct $\lambda^{!}$ that turns out to be "contracted" in comparison with λ, the true length of $P_0 P_\rho$ resting in S **:** $\lambda^{!} = \lambda / \gamma < \lambda$.. Similarly, the unprimed observer O compares the *primed* frame S''s stationary length $P_0' P_\rho' = \lambda'$ (Figure 3b) ***not*** with $P_0 P_\rho = \lambda$ (Figure 3a), but rather with λ''s computed update $P_0'^{\,!} P_\rho'^{\,!} = \lambda'^{\,!}$ (Figure 3c). Again, $P_0 P_\rho = \lambda$ is how he ***sights*** $P_0' P_\rho' = \lambda'$ from S with his actual eyes, but $P_0'^{\,!} P_\rho'^{\,!} = \lambda'^{\,!}$ is how his mind's eye pictures λ' as latterly *shifted* because of its host S''s rightward movement. Once more, $\lambda'^{\,!}$ proves to be contracted relatively to λ', the true length of $P_0' P_\rho'$ resting in S' **:** $\lambda'^{\,!} = \lambda' / \gamma < \lambda'$.

Note that the contracted lengths $\lambda'^{!} = P_0'^{!} P_\rho'^{!}$ and $\lambda^{!} = P_0^{!} P_\rho^{!}$ are ***never*** actually sighted. They are always products of O 's and O''s conscious computations. Yet the literature of special relativity was rife with uncritical stories of visually demonstrable length contractions until the middle of the last century. The mistake -- obvious enough in the approach we have followed -- was eventually spotted [13], of course, but only as late as in 1959.

10. Time dilations

Time dilations arise as a simple kinematical consequence of length contractions. Consider the coincidence $[P_0' : P_0]$ once more. This lasts for only a fleeting instant of time. Coming on a little later will be the event $[P_0' : P_\rho]$. We ask**:** What interval of *time* elapses *between* $[P_0' : P_0]$ and $[P_0' : P_\rho]$?

Suppose this time interval is τ as estimated in S, and τ' as estimated in S'. Then anyone resting in S figures out that, during the time τ, the site P_0' attached to S' has moved at the speed $c\beta$ from P_0 to P_ρ, a distance λ.

This leads to the conclusion $\tau = \lambda / (c\beta)$. On the other hand, an observer resting in S', say O', figures out that a specific line segment P_0P_ρ attached to the other frame S has been *moved **past*** P_0', again at the same speed $c\beta$. But, as we have seen, O''s measurements picture P_0P_ρ with a *contracted* instantaneous length $P_0'P_\rho' = \lambda' = \lambda / \gamma$. The time needed to move this length past P_0' is just $\tau' = \lambda / (c\beta\gamma) = \tau / \gamma < \tau$. This is the time dilation effect.

If a clock is attached to the site P_0' moving with the frame S', it records precisely the time τ' as lapsing between the events $[P_0' : P_0]$ and $[P_0' : P_\rho]$. What about τ then ? Notice that it is impossible for any *single* clock in the "stationary" frame S to record τ**!** Only a ***pair*** of synchronized clocks deployed at P_0 and P_ρ in S can record the lapse of the time τ between $[P_0' : P_0]$ and $[P_0' : P_\rho]$. And the τ so determined will be longer by the factor γ than τ', the interval sensed by the single "moving" S'-clock placed at P_0'.

The time dilation effect betrays no asymmetry between the frames S and S'. Readers can check this out by looking at the events $[P_\rho' : P_\rho]$ and $[P_0' : P_\rho]$ instead of $[P_0' : P_0]$ and $[P_0' : P_\rho]$. They should then find $\tau = \lambda'/(c\beta\gamma) < \tau' = \lambda'/(c\beta)$.

11. Aberration: The tunnel and the train revisited

On using polar coordinates via $x = r\cos\theta$ and $x' = r'\cos\theta'$, the relationship (21) of Sec.9 gives

$$r'/r = \gamma(1 + \beta\cos\theta), \qquad (22)$$

$$r/r' = \gamma(1 + \beta\cos\theta')\,; \qquad (23)$$

$$(1 + \beta\cos\theta)(1 + \beta\cos\theta') = 1/\gamma^2 = 1 - \beta^2. \qquad (24)$$

These express the well-known relativistic aberration effect in a manifestly symmetric from. Observe the important implication

$$\text{if } \theta = 0, \text{ then } \theta' = \pi, \quad \text{and} \quad \text{if } \theta = \pi, \text{ then } \theta' = 0. \qquad (25)$$

Results similar to Eq.(22) must hold for any other coincidence event $[P^{\bullet\prime} : P^{\bullet}]$ sighted at $[O' : O]$ concurrently with $[P' : P]$. From this readily follows

$$\frac{r' r^{\bullet}}{r \; r^{\bullet\prime}} = \frac{1 + \beta \cos \theta}{1 + \beta \cos \theta^{\bullet}}, \qquad (26)$$

where (as elsewhere in the future) variables tagged with the dot $(\bullet)$ refer to $[P^{\bullet\prime} : P^{\bullet}]$. We can now choose, with reference to our introductory example of the tunnel and the train, $[O' : O] \equiv$ [P1 : W4], $[P' : P] \equiv$ [P5 : W5] and $[P^{\bullet\prime} : P^{\bullet}] \equiv$ [P0 : W0]. This leads to

$$\theta = 0,\ \theta' = \pi;\quad r = \delta,\ r^{\bullet} = 4\delta;\quad r' = 4\delta',\ r^{\bullet\prime} = \delta',$$

δ and δ' being the inter-observer distances inside the tunnel and the train. Solving Eq.(26) with these inputs gives the train's speed as $\beta = 15/17$ times c -- a conclusion that could, of course, have been reached from a more direct one-dimensional analysis as well [14]. The corresponding length contraction factor turns out to be $1/\gamma = 8/17$. Variants of our story featuring $m+1$ equidistant watchmen and $n+1$ equidistant passengers with [P0 : W0] and $[\mathrm{P}_n : \mathrm{W}_m]$ sighted concurrently at $[\mathrm{P}_j : \mathrm{W}_i]$ can also be handled with equal ease.

With both $[P' : P]$ and $[P^{\bullet\prime} : P^{\bullet}]$ concurrently sighted at $[O' : O]$, Eqs.(20) and (21) of Sec.9 imply

$$r\, r^{\bullet} - x\, x^{\bullet} \;=\; r'\, r^{\bullet\prime} - x'\, x^{\bullet\prime}, \tag{27}$$

And with appropriate z and z' directions introduced in this context, it is also easy to prove [15]

$$y\, y^{\bullet} + z\, z^{\bullet} \;=\; y'\, y^{\bullet\prime} + z'\, z^{\bullet\prime}, \qquad [z = z' = 0]. \tag{28}$$

Let us now turn r, r' and $r^{\bullet}$, $r^{\bullet\prime}$ into vectors $\mathbf{r} = OP$, $\mathbf{r}^{\bullet} = OP^{\bullet}$, $\mathbf{r}' = O'P'$ and $\mathbf{r}^{\bullet\prime} = O'P^{\bullet\prime}$, with the scalar products

$$\mathbf{r}.\mathbf{r}^{\bullet} = r\, r^{\bullet} \cos\Phi, \qquad \mathbf{r}'.\,\mathbf{r}^{\bullet\prime} = r'\, r^{\bullet\prime} \cos\Phi'. \tag{29}$$

On subtracting Eq.(28) from Eq.(27) we finally obtain

$$r\, r^{\bullet}(1 - \cos\Phi) \;=\; r'\, r^{\bullet\prime}(1 - \cos\Phi'). \tag{30}$$

Incidentally, this expresses the invariance of a certain quantity known as the "spacetime interval" between the events $[P' : P]$ and $[P^{\bullet\prime} : P^{\bullet}]$. We shall have occasion to use Eq.(30) in Sec.14.

12. Composition of velocities

The mathematics of aberration subsumes the relativistic law of composition of velocities. Three inertial frames S, S' and S'' will figure in our discussion now. This makes the notation of primes incommoding, so we switch to superscripts (ρ) to signal the presence of ρ number of primes in any context ($\rho = 0,1,2$). The γ and β for the pair ($S^{(\rho)}$, $S^{(\sigma)}$) will be designated as $\gamma_{\rho\sigma}$ and $\beta_{\rho\sigma}$. Also, we turn $\beta_{\rho\sigma}$ into an $S^{(\rho)}$-resident vector $\boldsymbol{\beta}_{\rho\sigma}$ by attributing to it the direction of $S^{(\sigma)}$'s motion relative to $S^{(\rho)}$. With the triple coincidences $[P^{(0)} \mathbf{:} P^{(1)} \mathbf{:} P^{(2)}]$ and $[P^{\bullet(0)} \mathbf{:} P^{\bullet(1)} \mathbf{:} P^{\bullet(2)}]$ concurrently sighted at the cluster $[O^{(0)} \mathbf{:} O^{(1)} \mathbf{:} O^{(2)}]$, six other vectors of special interest to us will be $\mathbf{r}^{(\rho)} = O^{(\rho)} P^{(\rho)}$ and $\mathbf{r}^{\bullet(\rho)} = O^{(\rho)} P^{\bullet(\rho)}$. The angles $\theta_{\rho\sigma}$ and $\theta^{\bullet}{}_{\rho\sigma}$ corresponding to the erstwhile θ and θ' can be defined by

$$\boldsymbol{\beta}_{\rho\sigma} \,.\, \mathbf{r}^{(\rho)} = \beta_{\rho\sigma}\, r^{(\rho)} \cos\theta_{\rho\sigma}\,, \tag{31}$$

$$\boldsymbol{\beta}_{\rho\sigma} \,.\, \mathbf{r}^{\bullet(\rho)} = \beta_{\rho\sigma}\, r^{\bullet(\rho)} \cos\theta^{\bullet}{}_{\rho\sigma}\,. \tag{32}$$

An immediate consequence of Eq.(26) [Sec.11] in this context is

$$\frac{1 + (\cos\theta_{02})\,\beta_{02}}{1 + (\cos\theta^{\bullet}{}_{02})\,\beta_{02}} = \frac{1 + (\cos\theta_{01})\,\beta_{01}}{1 + (\cos\theta^{\bullet}{}_{02})\,\beta_{01}}\ \frac{1 + (\cos\theta_{12})\,\beta_{12}}{1 + (\cos\theta^{\bullet}{}_{12})\,\beta_{12}}\,. \tag{33}$$

In the case of *one-dimensional* motions, the events $[P^{(0)} : P^{(1)} : P^{(2)}]$ and $[P^{\bullet(0)} : P^{\bullet(1)} : P^{\bullet(2)}]$ can be chosen with *all* the $\theta_{\rho\sigma}$'s and $\theta^{\bullet}_{\rho\sigma}$'s taking on only two values, some 0 and some π, but always with $\theta^{\bullet}_{\rho\sigma} = \pi - \theta_{\rho\sigma}$ [cf. Sec.11, Remark (25)]. Then, with all the $\cos\theta_{\rho\sigma}$'s and $\cos\theta^{\bullet}_{\rho\sigma}$'s in Eq.(33) being just ± 1, this relationship turns out to be none other than the familiar law of composition of *one-dimensional* velocities.

Eq.(33) remains true even for the cases where motions span a second dimension, but proves inadequate then to completely determine $\boldsymbol{\beta}_{02}$. To plug this deficiency, we choose $[P^{(0)} : P^{(1)} : P^{(2)}]$ and $[P^{\bullet(0)} : P^{\bullet(1)} : P^{\bullet(2)}]$ in such a way that just $\theta_{01} = 0$ and $\theta^{\bullet}_{01} = \pi$. Then, on the one hand, $\theta_{10} = \pi$ as well as $\theta^{\bullet}_{10} = 0$ [cf. Sec.11, Remark (25)] **;** and, on the other, $\theta^{\bullet}_{02} = \pi - \theta_{02}$ together with (*how*?) $\theta^{\bullet}_{12} = \pi - \theta_{12}$. Hence by Eq. (22) [Sec.11],

$$r^{(1)}/r^{(0)} = \gamma_{01}(1+\beta_{01}),\quad r^{\bullet(1)}/r^{\bullet(0)} = \gamma_{01}(1-\beta_{01}),$$
$$r^{(2)}/r^{(1)} = \gamma_{12}(1+\beta_{12}\cos\theta_{12}),\quad r^{\bullet(2)}/r^{\bullet(1)} = \gamma_{12}(1-\beta_{12}\cos\theta_{12}),$$
$$r^{(2)}/r^{(0)} = \gamma_{02}(1+\beta_{02}\cos\theta_{02}),\quad r^{\bullet(2)}/r^{\bullet(0)} = \gamma_{02}(1-\beta_{02}\cos\theta_{02}). \qquad \textbf{(34-39)}$$

These imply

$$\gamma_{02}(1\pm\beta_{02}\cos\theta_{02}) = \gamma_{01}\gamma_{12}(1\pm\beta_{01})(1\pm\beta_{12}\cos\theta_{12}), \qquad (40)$$

which by addition and subtraction in turn lead to the general law of composition of velocities**:**

$$\gamma_{02} \; = \; \gamma_{01}\, \gamma_{12}\, (1 + \beta_{01}\, \beta_{12} \cos\, \theta_{12}\,) \, ; \qquad (41)$$

$$\beta_{02} \cos\, \theta_{02} \; = \; (\beta_{01} + \beta_{12} \cos\, \theta_{12}\,) \, / \, (1 + \beta_{01}\, \beta_{12} \cos\, \theta_{12}\,). \qquad (42)$$

13. The Doppler effect

The Doppler effect of optics can be interpreted as an offshoot of just *length* contractions. Imagine three inertial frames of reference $S^{(\rho)}$, (ρ = 0,1,2). In this section (but *not* elsewhere), we call $S^{(0)}$ "exotic", and the other two "ordinary". Two stationary wave profiles $\Lambda_{\kappa}{}^{(0)}$ are supposed to be imprinted along different directions in $S^{(0)}$, these directions being parallel to $c\,\boldsymbol{\beta}_{0\kappa}$ -- the $S^{(0)}$-relative velocities of the "ordinary" frames $S^{(\kappa)}$, (κ =1,2). The wavelengths λ featured in the two wave profiles are immense, and at the same time all equal to one another. To the observers residing in $S^{(\kappa)}$, however, these λ's appear *contracted* to $\lambda_{\kappa}{}^{!} = \lambda \, / \, \gamma_{0\kappa}$ [cf. Sec.9], whence, using Eq.(41),

$$\lambda_1{}^{!} \, / \, \lambda_2{}^{!} \; = \; \gamma_{02} \, / \, \gamma_{01} = \; \gamma_{12}\, (1 + \beta_{01}\, \beta_{12} \cos\, \theta_{12}\,). \qquad (43)$$

On holding $\beta_{12} < 1$ constant and passing into the limit as $\beta_{01} \rightarrow 1$, Eq.(43)

readily yields the Doppler formula [16]

$$\lambda_1^{\,!} / \lambda_2^{\,!} \;=\; \gamma_{12}\,(1 + \beta_{12} \cos \theta_{12}) = \;\gamma_{12}\,(1 - \beta_{12} \cos \theta^{\bullet}_{12}) \qquad (44)$$

Reassuringly, $\beta_{01} \to 1$ also implies $\beta_{02} \to 1$, as well as $\theta_{02} \to 0 = \theta_{01}$ in this context. Our limit thus reflects the speed of light for the (ultimately coinciding) pair of wave trains $\Lambda_\rho^{(0)}$ as measured from each of the subluminal ("ordinary") frames $S^{(\kappa)}$, ($\kappa = 1,2\,;\ \beta_{12} < 1$) **:** *the waves are electromagnetic* **!**

14. A trihedral angle and its varied looks

Lastly, we now need to consider an additional site $P^{\bullet\bullet(\rho)}$ in each frame $S^{(\rho)}$, so we superscribe all relevant symbols with (κ, ρ) to indicate the presence of κ dots ($\bullet$) and ρ primes ($'$) [$\kappa = 0,1,2\,;\ \rho = 0,1\,;\ \rho = 2$ is no more needed]. Assuming all the three coincidences [$P^{(\kappa\,,\,0)} : P^{(\kappa\,,\,1)}$] to be concurrently sighted at [$O^{(0)} : O^{(1)}$], we set the vectors $\mathbf{r}_\kappa^{\,(\rho)} = O^{(\rho)}P^{(\kappa\,,\,\rho)}$, and define the angles $\Phi_{\kappa\iota}^{\,(\rho)}$, $\theta_\kappa^{\,(0)}$ and the scalars μ_0, μ_1, μ_2 via

$$\mathbf{r}_\kappa^{\,(\rho)} \cdot \mathbf{r}_\iota^{\,(\rho)} \;=\; r_\kappa^{\,(\rho)}\, r_\kappa^{\,(\rho)} \cos \Phi_{\kappa\iota}^{\,(\rho)}\,; \qquad (45)$$

$$\boldsymbol{\beta} = \boldsymbol{\beta}_{01} = \;\; \mu_0\, \mathbf{r}_0^{\,(0)} + \mu_1\, \mathbf{r}_1^{\,(0)} + \mu_2\, \mathbf{r}_2^{\,(0)}\,; \qquad (46)$$

$$\boldsymbol{\beta} \cdot \mathbf{r}_\kappa^{\,(0)} = \;\beta\, r_\kappa^{\,(0)} \cos \theta_\kappa^{\,(0)}. \qquad (47)$$

Employing shortened subscripts 0≡12, 1≡20 and 2≡01, we also define

$$\vartheta_\kappa = \frac{\sin\frac{1}{2}\,\Phi_0^{(0)}\,\sin\frac{1}{2}\,\Phi_1^{(0)}\,\sin\frac{1}{2}\,\Phi_2^{(0)}\,\sin^2\frac{1}{2}\,\Phi_\kappa^{(1)}}{\sin\frac{1}{2}\,\Phi_0^{(1)}\,\sin\frac{1}{2}\,\Phi_1^{(1)}\,\sin\frac{1}{2}\,\Phi_2^{(1)}\,\sin^2\frac{1}{2}\,\Phi_\kappa^{(0)}} \ . \tag{48}$$

The μ's can be eliminated from (46) by using Eqs.(22), (29), (30), (45), (47) and (48). This eventually leads to a quadratic equation in $\gamma = (1-\beta^2)^{-1/2}$ **:**

$$\begin{vmatrix} \gamma^2-1 & \vartheta_0-\gamma & \vartheta_1-\gamma & \vartheta_2-\gamma \\ \vartheta_0-\gamma & 1 & \cos\Phi_2^{(0)} & \cos\Phi_1^{(0)} \\ \vartheta_1-\gamma & \cos\Phi_2^{(0)} & 1 & \cos\Phi_0^{(0)} \\ \vartheta_2-\gamma & \cos\Phi_1^{(0)} & \cos\Phi_0^{(0)} & 1 \end{vmatrix} = 0\,. \tag{49}$$

The angle triad $(\Phi_0^{(\rho)}, \Phi_1^{(\rho)}, \Phi_2^{(\rho)})$ defines a trihedral angle (also called spherical triangle) in $S^{(\rho)}$ ($\rho = 0,1$) if $\cos^2\Phi_0^{(\rho)} + \cos^2\Phi_1^{(\rho)} + \cos^2\Phi_2^{(\rho)} \le 1 + 2\cos\Phi_0^{(\rho)}\cos\Phi_1^{(\rho)}\cos\Phi_2^{(\rho)}$. It is possible to prove that this very condition also ensures the existence of real roots of (49) in the range $\gamma \ge 1$. The trihedral angle formed by the events $[P^{(\kappa,0)} : P^{(\kappa,1)}]$ can therefore look *arbitrarily* differing in its shape to $O^{(0)}$ and $O^{(1)}$. With the larger root γ, the sightings of $O^{(0)}$ and $O^{(1)}$ present the the circuit defined by $\kappa = 0\to1\to2\to0$ in *opposite* senses -- one clockwise and the other counterclockwise **!**

15. Conclusion

Einstein once remarks that the distinction between global and local times ‘fades away’ in our perceptions because we ‘fail to differentiate “simultaneously seen” from “simultaneously happening”’ [17] . Thus the space-time coordinates (*x*, *y*, *z*, *t*) often evoke in us the imagery of an actual eye *sighting* the position (*x*, *y*, *z*) from a distance *r* at the very time *t*. This naïve impression is *never* tenable in contexts of high precision like electrodynamics, (nor in astronomy), but beginners seldom find it easy to shake off. We have therefore formulated an overt "clock principle" at the outset to prepare them to accept *t* as a *computed* time rather than as a time of sighting. And then, with a further "camera principle" added in, we have not only described what different inertial observers in diverse states of motion actually ***see***, but provided an explicit briefing on what specific *inferences* special relativity expects them to draw from their sightings. This should help clear up many prevalent misconceptions, and pave the way for a deeper and more meaningful understanding of the kinematical basics of special relativity.

[1] In cases where *E* doesn't by itself alert *O* with electromagnetic signals, we can in principle imagine such alerting done by proxy, with the aid of instantly actuated ideal transducers placed in its vicinity.

[2] Albert Einstein, "On the electrodynamics of moving bodies". German original in Ann. Phys. **17**, 891-921 (1905) ; English translation in John Stachel, (Editor), *Einstein's Miraculous Year*, (Princeton U.P., Princeton, NJ, 1998), pp.123-160. An amusing quote from the then young Zurich professor struggling to make ends meet in 1909 appears in Philipp Frank, *Einstein*, *His Life and Times* [Jonathan Cape, London, (1953)], p.96**:** "In my relativity theory I set up a clock at every point in space, but in reality I find it difficult to provide even one clock in my room."

[3] Asserting *c*'s constancy ***across*** different inertial frames is not such a central issue as it is often made out to be. In point of fact, the latter-day definition of the meter (1983) as the vacuum path traversed by light in a specified fraction of a second has the effect of reducing *c*'s constancy to a glorious tautology. What really ***is*** essential for the logical development of special relativity is the ***isotropy*** of light's propagation in ***each*** given inertial frame *S*. This consists in the fact that, irrespective of the state of motion of its source, light propagates at one and the same speed in all *directions* relative to *S*.

[4] See, for example, the following **:**

-- R.Weinstein, "Observation of Length by a Single Observer", Am. J. Phys. **28**, 607-610, (1960) ;

-- G.D.Scott and M.R.Viner, "The geometrical appearance of large objects moving at relativistic speeds", Am. J. Phys. **33**, 534-536, (1965) ;

-- Arthur Komar, "Foundations of special relativity and the shape of the Big Dipper", Am. J. Phys. **33**, 1024-1027, (1965) ;

-- Asher Peres, "Relativistic Telemetry", Am. J. Phys. **55**, 516-519, (1987) ;

-- H.Blatter and T.Greber, "Aberration and Doppler shift: an uncommon way to relativity", Am. J. Phys. **56**, 333-338, (1988).

Most of the more recent publications in this context are apparently not meant to be *expository introductions* in the first place, and they do not therefore address the beginner's puzzlements anyway.

[5] A "light-minute" is the distance traversed by light signals in one minute. The speed of light is thus (tautologically) c = 1 light-min min^{-1}.

[6] B.Madhava, an undergraduate student, originally gave this argument (1980). Sadly, he ended his life from depression a couple of years later.

[7] The distances of the events [P0**:**W0] and [P5**:**W5] from [P1**:**W4] are vastly different as measured by W4 (Figure 1) and as measured by P1 (Figure 2). Such divergence in distance measurements is *not at all* uncommon ***even*** in

Newtonian kinematics. What sets special relativity apart is its radical view that not merely the *distances*, but even the ***time orderings*** of [P0**:**W0] and [P5**:**W5] must differ for P1 and W4.

[8] Since different inertial observers can place [P0:W0] and [P5:W5] in different time orderings under special relativity, these events are termed "spacelike separated". In contrast, "timelike separated" events would have an invariant time ordering relative to all inertial observers.

[9] The two distances r_ρ and r_ρ' need not be equal even under Newtonian kinematics (cf. Note 7 above).

[10] We note in passing, however, that *arbitrary* equidistance patterns such as equilateral triangles, squares etc violate this principle. The principle does not by any means purport to articulate an ***ontological*** necessity.

[11] A major advantage of this approach is that it altogether *bypasses* the problem of transporting a "standard meter stick" from S to S' with absolute certainty of its length remaining unaltered. (Alternative proposals, such as making use of some standard electromagnetic wavelength to measure *other* lengths, rely on extraneous knowledge not strictly needed in this context. They can even lead up to tautologies if handled improperly -- cf. Note 3 above.)

[12] The reason for negative signs appearing in the numerators of Eq.(4) is the *opposite* orientations ascribed to the x and the x' axes. This is done with a view to ensuring ***both*** $\beta > 0$ and $\beta' > 0$.

[13] Definitive references in this context are **:** J. Terrell, "Invisibility of the Lorentz contraction", Phys. Rev. **116**, 1041-1045, (1959), and R.Penrose, "The apparent shape of a relativistically moving sphere", Proc. Camb. Phil. Soc., **55**, 137- 139, (!959). For expository accounts, see, for example, V.S.Wiesskopf, "The Visual Appearance of Rapidly Moving Objects", Physics Today, **13** (9), 24-27, (1960) **;** and A. P. French, *Special Relativity*, (M.I.T. Introductory Physics Series, 1968), pp.149-152. Terrell's work includes additional insights not fully covered in our main text. The orthogonal triads $\mathrm{d}\mathbf{r} \equiv (\mathrm{d}r, r\,\mathrm{d}\theta, \mathrm{d}z)$ and $\mathrm{d}\mathbf{r}' \equiv (\mathrm{d}r', r'\,\mathrm{d}\theta', \mathrm{d}z')$ centered on $\mathbf{r}$ and $\mathbf{r}'$ in the frames S and S' have $\mathrm{d}z = \pm\,\mathrm{d}z'$ (cf. Note 15 below), while our aberration formulas (22-24) ensure $r\,\mathrm{d}\theta = (-r'\,\mathrm{d}\theta')$ as well. Accordingly, any pair of infinitesimal vectors $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\mathbf{r}'$ concurrently sighted in coincidence at $[O : O']$ will always have their ***projections*** in the planes perpendicular to the directions of $\mathbf{r}$ and $\mathbf{r}'$ precisely **congruent**. Identical cameras placed at $[O : O']$ and aimed in the directions of $\mathbf{r}$ and $\mathbf{r}'$ toward bundles of such infinitesimal vectors in S and S' therefore instantaneously capture images of exactly the same shape, but of sizes varying in the proportion r / r'. If the unprimed bundle $\{\mathrm{d}\mathbf{r}\}$ is imagined placed at $\mathbf{r}'$ in

S' instead of at **r** in S, it produces the same image in the S'-camera as the primed bundle $\{d\mathbf{r}'\}$ does, provided that $\{d\mathbf{r}\}$ is ***rotated*** around the z' direction in its new location by the angle $(\theta + \theta' - \pi)$. Penrose, on the other hand, draws attention to the curious circumstance that an S-resident ***sphere*** even when ***not*** infinitesimal still presents a perfectly circular outline to onlookers in S'. The layout of the individual sites on and inside the circular boundary appears distorted, though, as viewed from S' then **;** in particular, the center of the boundary and the center of the original sphere show up in *different* directions in S'.

[14] S.R.Madhu Rao, "Special Relativity -- An Exoteric Narrative", Resonance, (Bangalore), **3** (1), 61-66 ; **3** (5), 63-72 **;** Figure misprints corrected in **3** (7), 77 **;** (1998).

[15] With φ and φ' signifying obvious azimuthal angles, what corresponds to $y = y'$ [Eq.(17)] for the dotted event $[P^{\bullet\prime} : P^{\bullet}]$ is $y^{\bullet}/\cos\varphi = y^{\bullet\prime}/\cos\varphi'$. For this to ensure $y^{\bullet\prime} = y^{\bullet}$, it is necessary first to establish $\varphi = \pm\,\varphi'$. The last relationship follows because we must have $\varphi' = f(\varphi)$ and $\varphi = f(\varphi')$ [interchangeability of inertial frames], as well as $f(-\varphi') = -\,f(\varphi)$ [left-right symmetry with respect to the x-y plane]. The only analytic function that fits the bill is $f(\varphi) = \pm\,\varphi$, [which also gives, of course, $z^{\bullet\prime} = \pm\, z^{\bullet}$].

[16] Eqs.(**36**) and (44) together give $\lambda_1{}^! / \lambda_2{}^! = r^{(2)}/ r^{(1)}$, an implication J.Terrell calls "interesting" in his paper (p.1042, see Note 13 above). As hinted in a private communication received from B.Rothenstein, this relationship admits a simple interpretation in the context of Eq.(**36**)**:** It corroborates the relativistic invariance of the *number* of photons and of the *power* emanating from $[P^{(0)}\mathbf{:}\, P^{(1)}\mathbf{:}\, P^{(2)}]$. If N photons each of energy $hc/\lambda_\kappa{}^!$ have been emitted during the transit time $r^{(\kappa)}/c$, the power developed as estimated in the frame $S^{(\kappa)}$ is $(Nhc^2)/(r^{(\kappa)}\lambda_\kappa{}^!)$, [$\kappa$ = 1,2].

[17] Albert Einstein, "Physics and Reality" in *Ideas and Opinions*, (pp.290-323), (Crown Publishers,. New York, 1954, 1982), p.299.

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