GENERAL RELATIVISTIC EFFECTS ON QUANTUM INTERFERENCE

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Abstract

Using a novel approach, we work out the general relativistic effects on the quantum interference of de Broglie waves associated with thermal neutrons. The unified general formula is consistent with special relativistic results in the flat space limit. We also work out two examples, one in general relativity and the other in heterotic string theory, in order to obtain the first order correction terms. Measurement of these terms is closely related to the validity of the equivalence principle at a quantum level.

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The function of an expert is not to be more right than other people,
but to be wrong for more sophisticated reasons. – David Butler.

I. Introduction

Investigations of the gravitational effects on the quantum phenomena provide informations that could be useful in the development of a full theory of quantum gravity [1]. For instance, a semiclassical treatment reveals that quantum uncertainty in the source variables induces uncertainties in the metric components of gravity in a specific manner [2]. Another important phenomenon is the neutrino flavor oscillation induced by gravity [3-6] or by the recoil from virtual D-branes [7]. A different kind of theoretical approach in the calculations of the phase of quantum particles, neutrinos included, has yielded a very interesting result: The Dirac spin 1/2 particle has the exact covariant Stodolsky [8] phase $S/\hbar = (1/\hbar)\int p_\mu dx^\mu$ in a static gravity field [9]. Some of these investigations could provide appropriate theoretical backgrounds in the atmospheric neutrino experiments [10], or in the observations involving γ -ray bursts [11].

Recent interests also include the study of quantum interference fringes for thermal neutrons traveling along two different paths [12,13]. The interference with itself of the de Broglie wave associated with an ensemble of particles allows us to predict, via Huygen's principle, the motion of the wave. The presence of an external field modifies the motion and useful information about it could be gathered if one knows how the external field causes a shift in the interference fringes. A classic example is the Aharonov-Bohm (AB) fringe shift which provides information as to how the motion of electrons are modified in the presence of a magnetic potential [14]. The AB effect predicts the same gauge invariant fringe shift, viz., $\frac{e}{\hbar c_0}\oint A_\mu dx^\mu$ between the interfering beams both in special relativistic and nonrelativistic theories, where A_μ is the electromagnetic 4-potential, e is the electronic charge, $\hbar$ is Planck's constant and c_0 is the light speed in vacuum. However, the situation becomes more complicated, both experimentally and theoretically, when one introduces gravity and rotation. On the experimental side, the developments in the matterwave interferometry have shown a greater promise (over photon interferometry) in the measurements of the influences on the fringe shifts due to Earth's gravity and rotation. The neutron interferometry of the earlier experiments by Colella, Overhauser and Werner (COW experiment)[15] and an improved configuration of the later experiment by Werner, Staudenmann and Colella (WSC experiment) [16] provide information, at a classical level, of the effects of Earth's gravity and Coriolis force on the fringe shift.

On the theoretical side, a special relativistic treatment of the quantum fringe shift has been proposed by Anandan [13]. However, the effects of gravity and rotation are still considered in a separate way while using special relativistic expressions for energy and momentum. On the other hand, a more comprehensive analysis involving gravity and inertial forces requires the unified framework of general relativity that combines those forces into a single geometrical framework. The special relativistic effects should then follow as a limiting case. The current literature seems to still lack such a unified approach. This has been noted in a recent investigation by Zhang and Beesham [12]. However, they consider only Schawarzschild gravity and consequently the relativistic or the Coriolis parts do not appear in their formulation.

In this paper, adopting a new approach developed in a series of recent papers [17], we wish to work out the exact general relativistic equation for the quantum fringe shift for thermal neutrons. A "Coriolis" force appears *naturally* and the proposed equation can be applied to a fairly wide class of rotating metrics. The curved spacetime contributions to the special relativistic and Coriolis effects are displayed. As illustrations, we consider the Kerr metric of general relativity and the Kerr-Sen metric of heterotic string theory [18]. The later will enable us to also envisage the string effects.

The paper is organized as follows: In Sec.II, for ready reference, Anandan's special relativistic (SR) approach is summarized. In order to be reasonably exhaustive, we enumerate, in Sec.III, the salient features of the present approach leading to an appropriate form of the geodesic equation. In Sec.IV, the general relativistic equation for the quantum fringe shift is proposed. Two examples are considered in Sec.V, while some concluding remarks are given in Sec.VI.

II. Special relativistic quantum phase shift

In the literature, there exist several deductions of the individual parts of the special relativistic contributions to quantum phase shift [8,13,19,20]. However, using intuitive arguments, Anandan [13] proposed a general formula for the quantum phase shift based on a correspondence between the fringe shift and the classical special relativistic equations of motion. This remarkable general formula gives rise to all the above individual components. Consider two de Broglie wavelets originating at A and interfering at B, one traveling along ADB and the other along ACB, the plane ADBC being fixed to Earth and aligned vertically (fig.1). The vertical direction is determined by the resultant of gravity and the centrifugal force of Earth. In the absence of any external forces, BD=BC. The external field causes a shift in the position of B. Let $\tilde{v}$ be the velocity of the classical particle and $\delta\tilde{v}$ be the change in the tranverse velocity in the plane of interference. Then,

$$\Delta\phi_{SR} = \kappa d\frac{\delta\tilde{v}}{\tilde{v}} = \frac{\kappa d}{\tilde{v}}\frac{d\tilde{v}_{\perp}}{dt}\frac{l}{\tilde{v}} = \frac{\tilde{p}A}{\hbar\tilde{v}^{2}}\frac{d\tilde{v}_{\perp}}{dt} = \frac{\tilde{\gamma}mAd\tilde{v}_{\perp}/dt}{\tilde{v}\hbar} \tag{1}$$

where $\tilde{\gamma} = \left(1-\tilde{v}^{2}/c_{0}^{2}\right)^{-1/2}$, $A = ld$ is the planar area enclosed by the two paths of interfering beams, m is the neutron mass, $\tilde{p}$ is the momentum in flat space, and $\frac{d\tilde{v}_{\perp}}{dt}$ is the transverse component of acceleration in the plane ADBC. (We have made a slight change in the notation in order to be consistent with our later equations.) The formula (1) exhibits a general correspondence between the quantum phase shift and the classical equation of motion. The following cases were considered: (i) A particle with charge e moving in a magnetic field B. Then, $\tilde{\gamma}d\tilde{v}_{\perp}/dt = eB\tilde{v}/m$, which, when used in Eq.(1), immediately yields the AB effect, viz.,

$$\Delta\phi_{AB} = eBA/\hbar\,. \tag{2}$$

(ii) A spinless particle of mass m in a gravitational field, $d\tilde{v}_{\perp}/dt = -g$ where g is the gravitational acceleration on the surface of the Earth. Using the expression for the relativistic momentum $\tilde{\gamma}^2 = 1 + \tilde{p}^2/m^2c_0^2$ and the de Broglie relation $\tilde{p} = \hbar\tilde{\kappa}$, one obtains from Eq.(1), the following result:

$$\Delta\phi_{g.c.} = -gAm^2/\hbar^2\tilde{\kappa} - gA\tilde{\kappa}/c_0^2\,. \tag{3}$$

The COW experiment was accurate enough to measure the first quantum term, but not the second, that is, the so called relativistic term.

(iii) A particle moving in the Coriolis force field of Earth so that, to first order in Ω, $d\tilde{v}_{\perp}/dt = 2\left|\vec{\Omega}\times\vec{\tilde{v}}\right| = 2\Omega_n\tilde{v}$, where Ω_n is the component of Earth's angular velocity $\vec{\Omega}$ normal to the apparatus. Using the Planck-Einstein law $E/c_0 = \hbar\tilde{\omega} = mc_0\tilde{\gamma}$, one finds from Eq.(1) that

$$\Delta\phi_{Cor} = 2\tilde{\omega}\Omega_n A/c_0\,. \tag{4}$$

The WSC experiment has tested this effect to within a good accuracy.

(iv) This effect comes from the coupling of particle's spin to the background curvature. Again using (1) together with the Mathisson-Papapetrou force [21], the shift comes out to be

$$\Delta\phi_{s.c.} = -\frac{\hbar GMA\tilde{\omega}}{mc_0^3R^3}\,, \tag{5}$$

where G is the Newtonian gravitational constant, M is Earth's mass and R is the distance from the center. This effect is too tiny to be measurable at present. However, with all the above in view, we proceed to familiarize the readers with the salient features of our approach in the next section.

III. The approach: Basic equations

The basic idea of the method is to introduce the idea of an optical-mechanical analogy in general relativity [17]. The analogy provides an excellent tool that enables one to visualize the problems of geometrical optics as problems of classical mechanics and vice versa. The first step is to find out an optical refractive index n that is formally equivalent to the geometrized gravity field. This step in itself is not new. In the study of optical propagation in a gravity field, this index has been used in the literature *albeit* in an approximate form. For instance, the index equivalent to the exterior Schwarzschild field is usually taken to be $n \approx 1 + 2MG/rc_0^2$, where M is the central gravitating mass. In our approach, however, we consider the exact expression shown below.

Consider a static, spherically symmetric, but not necessarily vacuum, solution of general relativity written in isotropic coordinates

$$ds^2 = h(\vec{r})c_0^2 dt^2 - \Phi^{-2}(\vec{r})|d\vec{r}|^2 \tag{6}$$

where $\vec{r} \equiv (x, y, z) or (r, \theta, \varphi)$, and h, Φ could be the solution of Einstein's field equations. Many metrics of physical interest can be put into this isotropic form. The coordinate speed of light $c(\vec{r})$ is determined by the condition that the geodesic be null $(ds^2 = 0)$:

$$c(\vec{r}) = \left|\frac{d\vec{r}}{dt}\right| = c_0 \Phi \sqrt{h} \tag{7}$$

which immediately provides an effective index of refraction for light in the gravitational field given by

$$n(\vec{r}) = \frac{1}{\Phi\sqrt{h}}. \tag{8}$$

Henceforth, unless specifically restored, we take $G = c_0 = 1$. As the next step, we point out that the concept of the optical mechanical analogy can be used to recast the geodesic equation for *both* massive and massless particles into a single, exact Newtonian "F=ma" type of equation given by [17]

$$\frac{d^2\vec{r}}{dA^2} = \vec{\nabla}\left(\frac{1}{2}N^2\right), N(\vec{r}) = n(\vec{r})\sqrt{1 - \frac{m^2 h}{E^2}}, dA = \frac{dt}{n^2} \tag{9}$$

where m is the rest mass of the particle, E is the conserved total energy, $\vec{\nabla}$ is the gradient operator. All the standard geodesic equations in Schwarzschild gravity including some new insights in cosmology follow from the above equation. This remarkably simple feature of the geodesic equations is brought out by the use of the parameter A introduced by Evans and Rosenquist [22].

Alsing [23] has subsequently extended the method to broader class of metrics in general relativity and this work is going to provide the basic foundation of what follows. Consider the most general form of the metric given by

$$d\tau^2 = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j, i, j = 1,2,3, \tag{10}$$

Define the proper time $d\tau$, proper length dl and the velocity v measured with respect to this proper time as

$$v^i = \frac{dx^i}{d\tau}, \tag{11}$$

$$d\tau = \sqrt{h}(dt - g_i dx^i), v^2 = \gamma_{ij}v^i v^j, \tag{12}$$

$$dl^2 = \gamma_{ij} dx^i dx^j = \left(-g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}} \right) dx^i dx^j \ , \tag{13}$$

$$g_i = -\frac{g_{0i}}{g_{00}}, g^i = \gamma^{ij} g_j = -g^{0i}, . \tag{14}$$

The g_i and the proper velocity v_i are vectors defined in the 3-space characterized by the metric γ_{ij}. This metric is used to raise or lower the indices of the 3-vectors. Now, the metric (10) can be rewritten as

$$d\tau^2 = h(dt - g_i dx^i)^2 \left(1 - v^2\right). \tag{15}$$

The conserved energy E is given by

$$E = mg_{0\alpha} \frac{dx^\alpha}{d\tau} = \frac{m\sqrt{h}}{\sqrt{1-v^2}}. \tag{16}$$

The variational principle for the geodesics following from Eq.(15) is given by

$$\delta \int_{\vec{x}_1, t_1}^{\vec{x}_2, t_2} m d\tau = \delta \int_{\vec{x}_1, t_1}^{\vec{x}_2, t_2} L dt = \delta \int_{\vec{x}_1, t_1}^{\vec{x}_2, t_2} m\sqrt{h(\vec{r})} \sqrt{1 - v^2(\vec{r}, \tilde{v})} \tilde{\beta}(\vec{r}, \tilde{v}) dt = 0 \tag{17}$$

where

$$\tilde{\beta} = 1 - g_i \tilde{v}^i, \tilde{v}^i = \frac{dx^i}{dt}, v^i = \frac{\tilde{v}^i}{\tilde{\beta}\sqrt{h}}. \tag{18}$$

From the Lagrangian L, let us find the momenta conjugate to $\tilde{v}^i$, which is given by

$$\frac{\partial L}{\partial \tilde{v}^i} = E\left(g_i + \frac{v_i}{\sqrt{h}} \right). \tag{19}$$

Note that, since the vectors g_i and v_i are defined in a space with the metric γ_{ij}, we can identify the right hand side of the above as a vector in the same 3-space. Let us call it the momentum 3-vector

$$p_i \equiv E\left(g_i + \frac{v_i}{\sqrt{h}} \right). \tag{20}$$

It can also be verified that $H = \frac{\partial L}{\partial \tilde{v}^i} \tilde{v}^i - L \equiv p_i \tilde{v}^i - L = E$, which is a constant along the trajectory of a particle as stated in Eq.(16). Hence, it is possible to introduce Maupertuis principle

$\delta \int_{\vec{x}_1}^{\vec{x}_2} p_i \tilde{v}^i dt = 0$. Assuming further that the spatial part of the metric could be written in an isotropic form $dl = dl_E / \Phi, dl_E = \delta_{ij} dx^i dx^j, \gamma_{ij} = \delta_{ij} \Phi^{-2}$, we get $v^2 = n^2 \tilde{v}^2 / \tilde{\beta}^2$, and the Maupertuis variational principle yields, after some manipulations introducing the parameter A, the geodesic equation in the form

$$\frac{d^2\vec{r}}{dA^2} = \vec{\nabla}\left(\frac{n^2 v^2}{2}\right) + \frac{d\vec{r}}{dA} \times (\vec{\nabla} \times \vec{g}), \tag{21}$$

$$dA = n^{-2} \tilde{\beta} dt, \tag{22}$$

where $\vec{g} \equiv (g_i)$. This Newtonian form of the geodesic equation, valid for both massless and massive particles, has been obtained under the only assumption that the spatial part of the metric could be written in an isotropic form. On eliminating A from Eqs.(21) and (22), we can find the rotational correction terms to the well known Schwarzschild orbits for light and planets [23].

Equation similar to Eq.(21) has been derived in the literature [24], but only under the assumptions of small velocity and weak gravity. On the other hand, the novelty of the set of the above equations is that they hold for *all velocities as well as in strong gravity fields*. The Eq.(21) admits an immediate interpretation as describing the motion of a particle in a "potential" $(-1/2)n^2v^2$ and subjected to a "Coriolis" force $\frac{d\vec{r}}{dA} \times (\vec{\nabla} \times \vec{g})$, which would appear, for instance, in the absence of gravity $(n = 1, h = 1)$ in a coordinate system rotating with angular velocity $\vec{\Omega} = (1/2)(\vec{\nabla} \times \vec{g})$. Another advantage is that Eq.(21) is expressed in terms of the velocity measured in proper time in a rotating field and this is exactly the velocity measured by an observer comoving with the interferometer.

IV. General relativistic effect on the fringe shift

Our approach indicates that it is possible to simulate the quantum interference in Earth's gravity as an experiment in a rotating medium evidently described by n and other quantities that take care of the nonlinearities. Our task is to generalize formula (1) in terms of proper quantities, recalling that in a gravity field, one can measure only proper, and not coordinate, quantities. We make the reasonable assumption that n, h and Φ do not vary appreciably over the dimensions of the apparatus (a few centimeters). From Eq.(12), we get

$$d\tau = \tilde{\beta}\sqrt{h} dt, \tag{23}$$

and defining $d^2\vec{r}/d\tau^2 \equiv d\vec{v}_\perp / d\tau$, we can write, from Eq.(21), the transverse component of the proper acceleration as

$$\frac{d\vec{v}_\perp}{d\tau} = \left(\frac{1}{2n^2h}\right)\vec{\nabla}\left(v^2\right) + \left(\frac{2}{n^2\sqrt{h}}\right)\left(\vec{v}\times\vec{\Omega}\right) \tag{24}$$

where $\vec{\Omega} = \frac{1}{2}\left(\vec{\nabla}\times\vec{g}\right)$. Further, defining $p^2 = p_i p^i = \gamma_{ij} p^i p^j$, and introducing quantum relations, we have, from Eqs.(20) and (16), respectively,

$$p = \left[\frac{m\sqrt{h}}{\sqrt{1-v^2}}\right]\left[\frac{v^2}{h} + \frac{2g_i v^i}{\sqrt{h}} + g_i g^i\right]^{\frac{1}{2}} = \hbar\kappa \tag{25}$$

$$E = \frac{m\sqrt{h}}{\sqrt{1-v^2}} = \hbar\omega, \tag{26}$$

Reasoning along the lines similar to those in Ref.[13], and using Eqs.(24) and (25), we can write, under the assumption stated above, the general relativistic version of Eq.(1) as

$$\begin{aligned}\Delta\phi_{GR} &= \left(\frac{pA}{\hbar v^2}\right)\left(\frac{dv_\perp}{d\tau}\right) \\ &= \left[\left(\frac{1}{2n^2h}\right)\frac{d}{dr}\left(v^2\right) + \left(\frac{2}{n^2\sqrt{h}}\right)\left|\vec{v}\times\vec{\Omega}\right|\right]\left(\frac{pA}{\hbar v^2}\right) \\ &= \Delta\phi_1 + \Delta\phi_2,\end{aligned} \tag{27}$$

in which $dv_\perp / d\tau (\equiv d^2r/d\tau^2 - rd\varphi/d\tau)$ is the radial component of acceleration. The rotational effects are represented by g_i (through $\tilde{\beta}$), which occur in v^i [Eq.(18)], p [Eq.(25)] and, of course, in the Coriolis term $\left|\vec{v}\times\vec{\Omega}\right|$. Strictly speaking, even the area A above would no longer be planar in the gravity field. But considering the miniscule dimension of the apparatus compared to the radius of the Earth, we may disregard its departure from planarity in the practical computations. The equation (27) for the quantum fringe shift is the main result of our paper.

If one considers, as is done in Ref.[13], that gravity is given by the Newtonian law $v^2 = \left|d\vec{r}/dt\right|^2 = 2M/r$ in an otherwise flat space $(n=1, h=1)$ and that the rotational effects are contained exclusively in the classical Coriolis term so that $\tilde{\beta} = 1$, then one has

$$\Delta\phi_{SR} = \left[-g + 2\left|\vec{\tilde{v}} \times \vec{\Omega}\right|\right]\left(\frac{\tilde{p}A}{\hbar\tilde{v}^2}\right) \quad (28)$$

where

$$g = \frac{1}{2}\frac{d}{dr}(\tilde{v}^2) = M / R^2, \quad (29)$$

R being the radius of the Earth. One can see how beautifully the geodesic equation synthesizes into a single equation the effects of gravity and rotation that were considered as separate pieces in Ref.[13].

In this context, we note that Sakurai [25] presented an interesting derivation of the phase shift induced by Earth's rotation given by Eq.(4). It shows that the effect of Coriolis force can be transcribed as an AB-type effect. This transcription finds a natural place in our approach. It was already stated in the previous section that Eq.(21) exhibits a "Coriolis" type force. The equation can alternatively be regarded as a "Lorentz" force equation for the motion of a classical particle with charge to mass ratio q/m in a static classical electromagnetic field. By scaling the stepping parameter A to a variable having the dimensions of time t', and identifying the scalar potential $\psi(\vec{r}) = (1/2)n^2v^2(-q/m)^{-1}$ and the vector potential $\vec{A}(\vec{r}) = \vec{g}(q/m)^{-1}$, we have from the four potential $A^\mu = (\psi, \vec{A})$, the electric and magnetic fields as

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t'} - \vec{\nabla}\psi, \vec{B} = \vec{\nabla} \times \vec{A}. \quad (30)$$

The last equation can be rewritten as $q\vec{B} = 2m\vec{\Omega}$, where $\vec{\Omega} = (1/2)(\vec{\nabla} \times \vec{g})$ is the rotational velocity of the Earth with respect to distant stars [see Eq.(33) in the next section]. Now, it is well known that the AB phase difference in the interference region is proportional to the magnetic flux enclosed by the two paths as in Fig.1 and is given by the line integral over the two routes:

$$\phi_{ADB} - \phi_{ACB} = (q/\hbar)\oint \vec{A}.d\vec{l} = (q/\hbar)\int \vec{B}.d\vec{\sigma}. \quad (31)$$

Putting the value for $q\vec{B}$ in the above, we get $\phi_{ADB} - \phi_{ACB} = (2m/\hbar)\int \vec{\Omega}.d\vec{\sigma}$, where $d\vec{\sigma}$ is the oriented area of the parallelogram ADBC. We thus get just the expression (4) with the difference that our $\vec{\Omega}$ is completely determined by the metric 3-vector $\vec{g}$. One may also regard the fringe shift due to the Coriolis term as somewhat analogous to the quantized version of the Sagnac effect without the particle mass term [13].

V. Examples

(a) The Kerr solution

We shall first consider the Kerr solution although, probably, it does not adequately represent any realistic rotating source due to quadrupole considerations. Nonetheless, we can have an idea as to how the corrections appear in different terms. Since the apparatus is situated on the surface of the Earth, we can take the metric to lowest order in M/r:

$$d\tau^2\Big|_{\theta=\pi/2, M/r<<1} \cong \left(1-\frac{2M}{r}\right)\left[dt+\frac{2Ma}{r}d\varphi\right]^2-\left(1+\frac{2M}{r}\right)\left[dr^2+r^2d\varphi^2\right] \tag{32}$$

where r is the isotropic radial variable, $J \equiv Ma$ is the source angular momentum. Then one has the following expressions:

$$\Phi \cong 1-M/r, h \cong 1-2M/r, n \cong 1+2M/r, g_\varphi \cong -2Ma/r\ , \tag{33}$$

$$\left|\vec{\nabla}\times\vec{g}\right| = r^{-1}\frac{dg_\varphi}{dr} = \frac{2Ma}{r^3} = 2\left|\vec{\Omega}\right|. \tag{34}$$

Assuming the velocity to be v_0 at the asymptotic region so that $h=1$, we have $E/m = E' = 1/\sqrt{1-v_0^2}$, and using the expression for E, we find

$$v^2 = 1-h(1-v_0^2). \tag{35}$$

Since, to first order, $h \cong 1-2M/r$, we have, on Earth's surface,

$$(1/2)d(v^2)/dr = -g(1-v_0^2). \tag{36}$$

Here we encounter a similar situation that reminds us of the statements of Cardall and Fuller [26]: It is not possible to extract the pure gravitational field g in the locally inertial frame; there also appears a contribution gv_0^2 in Eq.(36) coming in from the asymptotic region. However, for the range of values $v_0 << 1,$ we can disregard this extra term for practical computations and take the usual Newtonian equation (29) so that $v \cong \tilde{v} = \sqrt{2M/r}$. Let us assess the terms, to lowest order, appearing in the second square bracket in Eq.(25). It follows from Eq.(21) that [23]

$$\tilde{v}^\varphi = d\varphi/dt \cong 2Ma/r^3, \tag{37}$$

so that we have

$$\tilde{\beta} = 1-g_i\tilde{v}^i = 1-g_\varphi\tilde{v}^\varphi \cong 1+4J^2/r^4, \tag{38}$$

$$\frac{2g_i v^i}{\sqrt{h}} \cong 2\left(1+\frac{M}{r}\right)\left[\frac{g_\varphi \tilde{v}^\varphi}{\tilde{\beta}}\right] \cong 8\left(1+\frac{M}{r}\right)\frac{J^2}{r^4}, \tag{39}$$

$$g_i g^i = g_\varphi g^\varphi = \left(\frac{2Ma}{r}\right) g^{0\varphi} \cong \frac{4J^2}{r^4}. \tag{40}$$

Therefore, to first order in (M/r), the last two terms in the second square bracket in the expression for p do not contribute, and hence, to this order, we may write: $p \cong \tilde{p} = \frac{m\tilde{v}}{\sqrt{1-\tilde{v}^2}} = \hbar\tilde{\kappa}$. Together with the expression $hn^2 = \Phi^{-2} \cong 1 + 2M/r$, and using the identity, $1+\frac{\tilde{p}^2}{m^2} = \frac{1}{1-\tilde{v}^2}$, we get from Eq.(27), the effects of only static gravity in terms of Φ:

$$\Delta\phi_1 = \Phi^2\left(-\frac{gAm^2}{\hbar^2\tilde{\kappa}}\right) - \Phi^2\left(gA\tilde{\kappa}/c_0^2\right). \tag{41}$$

The rotational effect is then contained in the Coriolis term *per se*. Writing $E = \hbar\tilde{\omega}\sqrt{h}$ and using it in $\Delta\phi_2$, we find that the term is modified also by the same factor

$$\Delta\phi_2 = \Phi^2(2A\tilde{\omega}\Omega_n/c_0). \tag{42}$$

Using the approximate dispersion relation in Eq.(37)

$$\tilde{\omega}^2 - \tilde{\kappa}^2 = \left(\frac{m^2}{\hbar^2}\right)\left(\frac{1}{1-\tilde{v}^2}\right)\left(h-\tilde{v}^2\right), \tag{43}$$

we get

$$\Delta\phi_2 = \Phi^2\left(\frac{2\Omega_n Am}{\hbar}\right) + \text{ small relativistic terms } O(c_0^{-2}). \tag{44}$$

Note that velocity dependent dispersion relations occur in a different context also in other works [7,27]. With the expression for Φ given in Eq.(33), we easily see the first order corrections to respective special relativistic effects. The importance of the correction factors will be discussed in the last section.

(b) The string theory

Low energy effective field theory describing heterotic string theory also produces rotating black hole solutions. A classical exact solution has been found by Sen [18] which we refer to here as the Kerr-Sen solution. The Kerr-Sen metric in the Einstein frame closely resembles the familiar Kerr solution in Boyer-Lindquist coordinates [28]:

$$d\tau^2 = \left(1-\frac{2M}{\rho}\right)dt^2 - \Sigma\left(\frac{d\rho^2}{\Delta}+d\theta^2\right) - \left[\rho(\rho+\xi)+a^2+\frac{2M\rho a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\varphi^2$$

$$+\frac{4M\rho a\sin^2\theta}{\Sigma}dtd\varphi, \qquad (45)$$

where the diltonic field Ψ, the electromagnetic potentials A_t, A_φ and the tensor gauge potential $B_{t\varphi}$ are given by

$$\Psi = -\ln\left[\frac{\Sigma}{\rho^2+a^2\cos^2\theta}\right], A_\varphi = -\frac{2\sqrt{2}a\rho Q\sin^2\theta}{\Sigma}, A_t = \frac{2\sqrt{2}\rho Q}{\Sigma}, B_{t\varphi} = \frac{a\rho Q^2\sin^2\theta}{M\Sigma},$$

$$\Sigma = \rho(\rho+\xi)+a^2\cos^2\theta, \Delta = \rho(\rho+\xi)+a^2-2M\rho, \xi = Q^2/M. \qquad (46)$$

The metric describes a black hole with mass M, dilatonic charge Q, angular momentum aM, and magnetic dipole moment aQ. The quantity ξ has the dimension of length. For $Q=0$, the metric reduces to the Kerr solution of general relativity and for $a=0$, it reduces to the Gibbons-Garfinkle-Horowitz-Strominger (GGHS) black hole solution [29] with the redefinition $\rho \to \rho - \xi$.

Under the weak field assumption as in (a), we have, in the equatorial plane,

$$d\tau^2\Big|_{\theta=\pi/2, M/(\rho+\xi)<<1} \cong \left(1-\frac{2M}{\rho+\xi}\right)\left[dt+\frac{2Ma}{\rho+\xi}d\varphi\right]^2 - \left(1+\frac{2M}{\rho+\xi}\right)d\rho^2 - \rho(\rho+\xi)d\varphi^2. \qquad (47)$$

Under the reasonable assumption that $\xi << \rho$, we can write $\rho(\rho+\xi) \cong (\rho+\xi)^2$. In this case, we can adopt the isotropic radial variable r related to the standard variable ρ via $r = (\rho+\xi)e^{-M/(\rho+\xi)}$ and retaining terms in lowest order in $M/(\rho+\xi)$, we have $r \cong \rho+\xi-M$. The metric for large r now has the form

$$d\tau^2\Big|_{\theta=\pi/2, M/(r+\xi)<<1} \cong \left(1-\frac{2M}{r+\xi}\right)\left[dt+\frac{2Ma}{r+\xi}d\varphi\right]^2 - \left(1+\frac{2M}{r+\xi}\right)[dr^2+(r+\xi)^2d\varphi^2]. \qquad (48)$$

This has exactly the same form as Eq.(32) under the identification $r \to r+\xi$ and all the calculations in (a) can be carried out in a similar manner. We have

$$\Phi^2 \cong 1 - \frac{2M}{r} + \frac{2M(\xi + M)}{r^2} . \qquad (49)$$

Thus we see that the effect of the dilaton ξ appears only in the order $O(r^{-2})$ and its measurement appears to be beyond the present capabilities.

VI.Concluding remarks

The importance of the present work lies in its relation to the (weak) principle of equivalence. If a viable theory of quantum gravity has to be formulated, the validity of this principle must be tested even at a quantum mechanical level. Anandan [13] suggested a method which involves carrying out the experiment with the interferometer at a distance r from the axis of rotation. If the principle is valid quantum mechanically, one should obtain the phase shift Eq.(3) with $g = \Omega^2 r$. On the other hand, from the classical point of view, it is well known that the principle is embedded in Einstein's theory of general relativity in the form of geodesic equations which follow from the Bianchi identities. Hence, a natural question arises if we can write the geodesic equation in a form that at once reveals the effect of gravity and rotation on the quantum fringe shift while, at the same time, leads to corresponding special relativistic results in the limit. Our analysis above shows that it is indeed possible and a unified expression for the general relativistic quantum fringe shift was proposed. Thus, a test of the principle in the quantum regime involves looking for general relativistic correction terms in an interferometric experiment. It does no longer seem necessary to regard the equivalence principle as a mere equality of Newtonian forces, viz., $g = \Omega^2 r$, since general relativity encapsules something more in its nonlinearities than just this. The interesting analogy between the Coriolis and the AB effect has also been pointed out.

We had worked out two examples which reveal first order corrections $\frac{2M}{R}\left(\frac{gAm^2}{\hbar^2\tilde{\kappa}}\right)$, $\frac{2M}{R}(gA\tilde{\kappa}/c_0^2)$ and $\frac{2M}{R}\left(\frac{2\Omega_n Am}{\hbar}\right)$ that could be measured in the COW or WSC-type experiments. These corrections are also consistent from an intuitive point of view, although they arose here from an elaborate analysis. Since $2MG/Rc_0^2 \approx 10^{-9}$, measurements of these corrections in an Earth bound configuration require a sensitivity at least of the order of $\sim 10^9$ times more than what the present setup offers [30]. The string corrections appear only in the second order in the form $2Q^2/R^2$ and although it is an order of magnitude larger than the spin-curvature term $O(R^{-3})$, Eq.(5), it still looks fairly out of question at present.

As a final comment, we wish to point out that the validity of the principle of equivalence for *charged* particles is still a debatable issue [31]. At a fundamental level, the issue relates to the question as to how the (radiation) energy is to be defined: Is it a conserved quantity associated with Lorentz boosts in Minkowski space [32] or with the time translations [31]? The resolution of this question has a deep relevance to whether the breakdown of Lorentz invariance could be interpreted as a violation of the principle of equivalence, although a positive interpretation is commonly adopted in the literature [7,33]. These issues will be addressed in a future communication.

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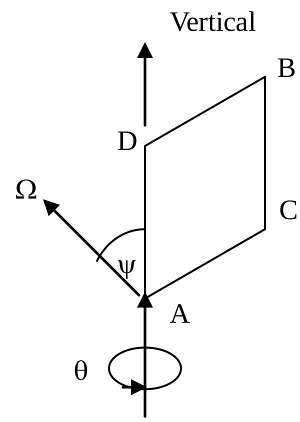

FIG. 1. Experiment to measure the phase shift due to earth's rotation in neutron interference. The interfering beams travel along paths ADB and ACB in a neutron interferometer. $\vec{\Omega}$ is the earth's angular velocity: $\Omega_n = \Omega \sin\psi \sin\theta$.