IS THE SPACE NON-EUCLIDEAN IN GENERAL RELATIVITY?

G.H.KESWANI

Abstract

We show that if the 3-space is non-Euclidean according to Einstein[1], $g_{ij}\, dx_i\, dx_j$ defining the metric of space-time, the perihelionic advance of Mercury in its orbit in one terrestrial century will not be 43 arc-sec.

When the prevailing geometry is taken as non-Euclidean, the ratio circumference/radius is not 2π, which is taken in the calculation of the figure of 43 arc-sec. It is in fact less than 2π per revolution, the cumulative effect being 65 arc-sec in the same period of hundred earth-years. It is not the measure of the circumference of the orbit, which is affected, but the radial measure and the related angle.

The equation of the orbit of Mercury is obtained from the exterior Schwarzschild metric:

$$ds^2 = (1\text{-}2m/r)dt^2\text{-}[(1\text{-}2m/r)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varnothing^{\,2})]/c^2, \qquad \textbf{(1)}$$

regarding the Sun as a point-mass.

Various symbols have the usual meaning, r being the curvature coordinate, so that $4\pi r^2$ is the invariant surface area of the spherical surface centred on the spherical mass. r is not the geodetic radial distance which is $\int dr/(1\text{-}2m/r)^{1/2}$ exterior to the Sun, and $\int dr/(1\text{-}2m^2/R^3)^{1/2}$ inside the Sun[2]. As usual, the constant radius, $R_\odot$ of the sun and the average radius, R, of the orbit of Mercury are assumed. The numerical values of these two are approximations in a Euclidean space but this will have little effect on the accuracy necessary to the present purpose.

Schutz[3] has given an elegant analysis to get the usual figure of advance of the perihelion per revolution, assuming Schwarzschild exterior metric throughout, i.e., for a point-mass which is,

$$\Delta \varnothing = 2\pi(1\text{-}6m^2/L^2)^{-1/2},$$

where, the angular momentum, L, is given by the equation:

$$L^2 = mr / (1\text{-}3m/R).$$

To a good approximation, the advance from one perihelion to the next is then:

$$\Delta\varnothing \sim 6\pi\, m/R \text{ radian.}$$

R the average distance of Mercury from the Sun is taken by Schutz as 5.55×10^7 km, which is more correctly, 5.79×10^7 km. Likewise, Schutz takes m= Sun's mass x $G/c^2 = 1.47$ km, which is more nearly equal to 1.4767 or 1.48 km. $R_\odot = 6.96 \times 10^6$ km.$M_\odot = 1.9891 \times 10^{30}$ kg.

Thus, Schutz gets the famous figure of advance of Mercury's perihelion per terrestrial century, of 43 arc-sec.

It is well-established that the observed advance of the perihelion of Mercury, not explicable by Newtonian theories, is ~ 43 arc-sec in one terrestrial century. This is equal to 4.99 x 10^{-7} radian per orbit. This has been verified from reliable astronomical observations now over very long periods of time since the 19^{th} century. The question I raise is: What does the General Theory say?

Geometry of Orbital Space

The theory of General Relativity is involved in two ways: (i) in the equation of the orbital motion, and (ii) in the geometry of Mercury's orbital space, also, according to the same theory.

In literature the second factor is not considered as significant and is omitted altogether. The ratio, circumference/radius, in a non-Euclidean space will not be 2π. Indeed R.C. Tolman[4] and G.C. McVittie[5], both considered the fact of non-Euclidean geometry but concluded that no significant error is introduced by ignoring this fact. They argued that whether one takes (i) r, or $\int dr/ (1-2m/r)^{1/2}$, or (ii) s, as the proper time at Mercury, or (iii) t as coordinate time at the origin (or, for that matter the proper time measured by an astronomer on the earth) in the calculation, the end result is not sensibly affected. L.D.Landau and E.M.Lifshitz[6] said, that following the GR (Schwarzschild metric), the ratio of the circumference ($2\pi r$) to the radius $_0\int^r \sqrt{g_{11}}.\ dr$, is less than 2π. But they did not numerically calculate the cumulative difference in the angle, in one terrestrial century. The fly in the ointment is $\varnothing$, the angle swept out in the orbit of Mercury. Also one should consider the interior metric within the Sun for a more accurate result. McVittie integrates from $R_\odot$ to R, not zero to R.

Quantitatively, as a result of the small change (increase) in the radial distance, per se (there is no circumferential change as can be seen from the metric), in non-Euclidean space there will be a small reduction (in the angle swept by Mercury) of –7.618 x 10^{-7} radian, in one revolution of Mercury. (Calculation given in the sequel.)

The change gets enhanced by a multiplier of 100 (terrestrial years), in which the faster Mercury with an orbital period of 87.97 Mercury-days, compared to 365.26 earth-days in one terrestrial year results in a multiplier equal to 100 years x 365.26 / 87.97 = 415.2 Mercury-years, i.e., its orbital revolutions, counted from perihelion to the next successive perihelion.

Therefore, in one terrestrial century, Mercury makes 415.21 revolutions – more than 4-times faster than the earth's - and a noticeable reduction (not advance) in the angle, cumulatively takes place as given by the following calculation:

Reduction from 2π in one revolution of Mercury due to non-Euclidean space:

-7.618 x 10^{-7} radian x $(180/\pi)$	[Conversion from radian to degree]
x 3600	[Conversion from degree to arc-sec, 60min. x 60sec.]
x 415.2	[Number of revolutions of Mercury in 100 earth-years]

= - 65.24 arc-sec, which is the angular reduction in the position of the perihelion of the planet Mercury per terrestrial century, due to non-Euclidean geometry of space where the ratio (circumference / radius is less than 2π), the angle from perihelion to the next perihelion being 2π radians – 65.24 arc-sec. Identical calculations are done for the advance of the perihelion except that the advance is + 4.99 (~5) x 10^{-7} radian in one revolution of Mercury, as against –7.618 x 10^{-7} radian per orbit for the non-Euclidean space.

The Main Issue

The question that now arises is: What is the solution? Answer: It is possible to maintain the figure of advance of 43 arc-sec per terrestrial century, if $g_{ij's}$ are regarded like potentials, physically, but not affecting space geometry which remains Euclidean. We have to come to grips with the matter.

Indeed, years ago in 1939-1940, in a general context (perihelionic advance of Mercury was not in his picture), Prof.N.Rosen (of MIT) had made an identical proposal in two papers, abstracts from which are reproduced below.

ABSTRACTS

JANUARY 15, 1940 PHYSICAL REVIEW VOLUME 57

General Relativity and Flat Space. I

N.ROSEN

Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received October 30, 1939)

Within the framework of the general theory of relativity, it is proposed to introduce at each point of space-time a Euclidean metric tensor* $\gamma_{\mu\nu}$ in addition to the usual Riemannian metric tensor $g_{\mu\nu}$. In this way one imparts tensor character to quantities which in the usual form of the theory do not have it. For example, one can obtain a gravitational energy-momentum density tensor in place of the usual pseudo-tensor. Furthermore, one can impose four additional covariant conditions on the gravitational field and thus restrict the form of the solution for the field corresponding to a given physical situation.

P. 147

JANUARY 15, 1940 PHYSICAL REVIEW VOLUME 57

General Relativity and Flat Space. II

N.ROSEN

Massachusetts Institute of Technology, Cambridge, Massachusetts

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The possibility is considered of interpreting the formalism of the general theory of relativity in terms of flat space, the fundamental tensor $g_{\mu\nu}$ being regarded as describing the gravitational field but having no direct connection with geometry.** The resulting theory in general leads to the same predictions as the Einstein theory, but there are cases where the predictions differ. The present theory may explain the principal results obtained by D.C. Miller in his “ether-drift” experiments. The implications of the theory for cosmology are being touched upon.

P. 150

* Incidentally, the word tensor was coined by W.Voigt in 1887 and he used it in elasticity-theory, in which stresses and strains of various kind and magnitude could arise in different directions at one point like g_{ij} in relativity.

** Rosen is clear here. The present author considers it necessary to adopt this idea, to rescue the figure of 43 arc-sec, which is well established by astronomers. If the geometry is non-

Euclidean, it will affect the measure of the position of the point of the perihelion of Mercury in its orbit, theoretically.

Rosen had been Einstein's collaborator and they wrote two papers together in 1935, Podolski being the third partner in one paper – a rare honour. EPR paper is famous and debate on it has not yet ended. But Einstein's response to the 1939/1940 paper of Rosen is not known.

It was a favourite theme, almost a creed, of Einstein that the geometry of space in a gravitational field was non-Euclidean, right upto his last book of 1950. (See Reference 1.)

Calculation of deficit in angular measure due to non-Euclidean Space

We now calculate the ratio circumference/radius. In principle, we only carry McVittie's and Landau's calculation forward. G.C.McVittie correctly worked out the difference as $(8 \times 10^{+6})^{-1}$ in the radius for a much closer distance $R = 2.48R_\odot$, very close to the Sun (actually, $R = 8.3R_\odot$) but observed that "identification of r with the heliocentric distances computed by astronomers on the hypothesis of Euclidean geometry introduces an error no larger than one part in eight millions," and just neglected it. Actually, the change in the angle, which accumulates with time, and not in the radius, which is significant. A large multiplier over a century applies, yielding a substantial cumulative figure. Not only the exterior metric but also the interior metric for the Sun, will be considered, although this makes a smaller contribution of about 7% of the total of both.

Using the interior and exterior Schwarzschild metrics[2], we have the radial geodesic distance:

$$\rho = \int_0^{R_\odot} (1-2mr^2/R_\odot^3)^{-1/2} dr + \int_{R_\odot}^{R} (1-2m/r)^{-1/2} dr, \qquad \textbf{(2)}$$

where, $R_\odot$ = radius of curvature at the surface of the Sun (taken as bounded), equal to 0.696×10^6 km and, R= radius of curvature at the mean position of Mercury with reference to the origin (centre of the Sun) equal to 5.79×10^7 km, quoted earlier.

$R_\odot$ and R can, without sacrifice of the necessary accuracy, be identified with the astronomer's measures of the radius of the Sun (its surface is fuzzy) and the average distance of the centre of Mercury (its orbit is elliptical) from the centre of the Sun, respectively.
Expanding (2) we get

$$\rho = \int_0^{R_\odot} (1+mr^2/R_\odot^3 + \ldots)\, dr + \int_{R_\odot}^{R} (1+ m/r + \ldots)\, dr \sim [R + m(1/3^* + \log_e{}^{**} R/R_\odot)] \qquad (3)$$

to first order which suffices.

* m/3 in the interior field.

** $m \log_e R/R_\odot$ in the exterior field.

Then, the angle swept per revolution would be less than 2π and would be equal to:

$$2\pi R/\rho = 2\pi R / [R+m(1/3 + \log_e R / R_\odot)] \sim 2\pi - 7.618 \times 10^{-7} \text{ radian.}$$

The second term represents departure from Euclidean geometry. And in one terrestrial century, the deficit of 7.618×10^{-7} radian per revolution, amounts to ~65 arc-sec.
See the calculation at the end.

The matter is important, and needs discussion in a Journal or Academy. I should say, all the more, because the shifts of the periastron in the binary pulsars are very large, by many orders of magnitudes, above Mercury's advance. Is the dynamics of motion of these pulsars being played out in a Euclidean space? We shall deal with this matter separately. The dynamics of the binary pulsars are rather complex, involving gravitational radiation, etc.

My purpose was to bring these fundamental matters into focus.

CALCULATION:

$2\pi R/[R + m (1/3 + \log_e R/R_\odot)]$** $R/R_\odot = \frac{5.79 \times 10^7}{0.696 \times 10^6} = 83.19$

$=2\pi/[1+m(1/3 + 4.4212)/R]$

$\sim 2\pi[1-(1.4767 \times 4.754)/5.79 \times 10^7]$

$\sim 2\pi - 2\pi \times 7.020/5.79 \times 10^7$

$\sim 2\pi - 7.618 \times 10^{-7}$ radian in one revolution.

In one terrestrial century, Mercury makes 365.26 x 100/87.97 = 415.21 revolutions. Then the deficit in the angle swept in one revolution would be 415.2 x (180/π) x 3600 x 7.618 x 10^{-7}, which works out at 65.24 or ~65 arc-sec per terrestrial century.

NOTE: We have not considered here the effect of the interior metric on the dynamics of the orbital motion, only exterior metric of empty space for a point-mass, is assumed to prevail in the usual calculation of the trajectories of Mercury.

** The contribution by the interior metric, (1/3), compared to the exterior one, ($\log_e R/R_\odot$) , is (1/3) : (4.212) = 1/12.636, i.e., 7.3% of the total.

1. A.Einstein, The books: The Principle of Relativity, Dover, p.161; The Meaning of Relativity, Methuen, 1950 p.59. In the first, Einstein said, "Euclidean geometry does not hold even to a first approximation," and in the second, "In the presence of a gravitational field, the geometry is not Euclidean."
2. See: J.L.Synge, Relativity: The General Theory, North-Holland Pub. Co., 1965, pp.275-289.
3. B.F.Schutz, A First Course in General Relativity, Cambridge Univ. Pr., 1992, pp.283-84. The more accurate figures are: the average distance of Mercury from the Sun, $R = 5.79 \times 10^7$km, m=1.4767 km and $R_\odot=0.696 \times 10^6$km. These latter figures have been used in the calculations below. Schutz gives the figure of Mercury's advance 4.99 x 10^{-7} radians per orbit, and of the period 0.24 earth years.
4. R.C.Tolman, Relativity Thermodynamics and Cosmology, Oxford, 1949 p.208, footnote.
5. G.C.Mc Vittie, General Relativity and Cosmology, Chapman&Hall, 1956, pp.87-88.
6. L.D.Landau and E.M.Lifshitz, The Classical Theory of Fields, Pergamon, 1959,p.307.
