On Obtaining a Quantum-Mechanical Time Operator.

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Quantum mechanics is based on a series of postulates, which basically state that reality is wave-like in nature; this is inherent in the formulation of Ψ, the state function. When Ψ is solved according to the constraints of the time-dependent Schrodinger equation, this leads to the formulation of certain stationary states, ψ, which are time-independent, according to the following equation:

(1) $\Psi = e^{-iEt/\hbar}\,\psi$.

If $\Psi = f + ig$, where f and g are real-valued functions, and i is the square root of minus one, then the complex conjugate of Ψ is $\Psi^* = f - ig$.

For a stationary state, independent of t, the following relations hold:

(2) $|\Psi|^2 = |f\ \psi|^2 = (f\ \psi)^*\ (f\ \psi) = (f)^*\ (\psi)^*\ f\ \psi$.

If we take f to be $e^{-iEt/\hbar}$ (in the time-independent Schrodinger equation) then

(3) $e^{iEt/\hbar}\,\psi^*\ e^{-iEt/\hbar}\,\psi = e^0\ \psi^*\ \psi = |\psi|^2$.

Hence, $|\Psi|^2 = |\psi|^2$. However, for those states functions whose stationary states do not depend on i, let us propose an alternate way of fulfilling this condition. Let us not consider the complex conjugate, since the time-independent state function does not depend on i. For this, we need then that the following condition hold:

(4) $|\Psi|^2 = |e^{-iEt/\hbar}\ \psi\ e^{-iEt/\hbar}\ \psi| = |e^{-iEt/\hbar}\ e^{-iEt/\hbar}\ \psi\ \psi| = |e^{-iEt/\hbar}\ e^{-iEt/\hbar}|\ |\psi\ \psi| = |\psi|^2$.

For this, we need then to fulfill the following relation:

(5) $|(e^{-iEt/\hbar})^2| = 1$.

Relation 5 is then equivalent to saying the following:

(6) $(e^{-iEt/\hbar})^2 = 1$ or $(e^{-iEt/\hbar})^2 = -1$.

Relation 6 then leads us to the following relation:

(7) $e^{-iEt/\hbar} = 1$ or $e^{-iEt/\hbar} = -1$ or $e^{-iEt/\hbar} = i$ or $e^{-iEt/\hbar} = -i$.

If we remember Euler's relation ($e^{ix} = \cos x + i \sin x$), then relation 7 can be formally broken up into the following four relations, which hold alternatively, strictly one at a time:

(8) $e^{-iEt/\hbar} = \cos(-Et/\hbar) + i \sin(-Et/\hbar) = 1$; or

(9) $e^{-iEt/\hbar} = \cos(-Et/\hbar) + i \sin(-Et/\hbar) = -i$; or

(10) $e^{-iEt/\hbar} = \cos(-Et/\hbar) + i \sin(-Et/\hbar) = -1$; or

(11) $e^{-iEt/\hbar} = \cos(-Et/\hbar) + i \sin(-Et/\hbar) = i$.

In order to solve relation 8, we need an argument that makes cos x = 1 and sin x = 0. For practical purposes, let us limit our x to be 2π >= x >= -2π . There is the trivial solution, $x_1 = 0$. There are two other solutions which we will save for the end of our derivation.

From $x_1 = 0$ we can obtain t_1. If cos (0) + i sin (0) = 1 + 0 = 1 (relation 8) holds, then it follows that $-Et/\hbar = 0$. If $E \neq E_0 \neq 0$ and $\hbar \neq 0$, then it follows that t = 0. We will label this t to be t_1, as follows:

(12) $t_1 = 0$.

Let us proceed with relation 9. In order to solve relation 9, we need an argument that makes cos x = 0 and sin x = -1. There are two solutions, x_2.and x_3. Let $x_2 = -\pi/2$ and $x_3 = 3\pi/2$.

From x_2 we can obtain t_2. If cos (-π/2) + i sin (-π/2) = 0 + -i = -i (relation 9) holds, then it follows that $-Et/\hbar = (-\pi/2)$. From this, we can see that:

(13) $t = (\pi\hbar)/2E$.

We will label this t as t_2. If $\hbar$ is defined as h/(2π) then:

(14) $t_2 = (\pi h)/(4\pi E) = h/4E$.

Similarly, from x_3 we can obtain t_3. If cos (3π/2) + i sin (3π/2) = 0 + -i = -i (relation 9) holds, then it follows that $-Et/\hbar = (3\pi/2)$. From this, we can see that:

(15) $t = (-3\pi\hbar)/2E$.

We will label this t as t_3. If $\hbar$ is defined as h/(2π) then:

(16) $t_3 = (-3\pi h)/(4\pi E) = (-3h)/4E$.

Let us proceed with relation 10. In order to solve relation 10, we need an argument that makes cos x = -1 and sin x = 0. There are two solutions, x_4.and x_5. Let $x_4 = -\pi$ and $x_5 = \pi$.

From x_4 we can obtain t_4. If cos -π + i sin -π = -1 + 0 = -1 (relation 10) holds, then it follows that $-Et/\hbar = -\pi$.

From this, we can see that:

(17) $t = (\pi\hbar)/E$.

We will label this t as t_4. If $\hbar$ is defined as h/(2π) then:

(18) $t_4 = (\pi h)/(2\pi E) = h/2E$.

Similarly, from x_5 we can obtain t_5. If cos π + i sin π = -1 + 0 = -1 (relation 10) holds, then it follows that $-Et/\hbar = \pi$.

From this, we can see that:

(19) $t = (-\pi\hbar)/E$.

We will label this t as t_5. If $\hbar$ is defined as h/(2π) then:

(20) $t_5 = (-\pi h)/(2\pi E) = -h/2E$.

Let us proceed with relation 11. In order to solve relation 11, we need an argument that makes cos x = 0 and sin x = 1. There are two solutions, x_6.and x_7. Let $x_6 = -3\pi/2$ and $x_7 = \pi/2$.

From x_6 we can obtain t_6. If cos (-3π/2) + i sin (-3π/2) = 0 + -i = -i (relation 11) holds, then it follows that $-Et/\hbar = -3\pi/2$. From this, we can see that:

(21) $t = (3\pi\hbar)/2E$.

We will label this t as t_6. If $\hbar$ is defined as h/(2π) then:

(22) $t_6 = (3\pi h)/(4\pi E) = 3h/4E$.

Similarly, from x_7 we can obtain t_7. If cos (π/2) + i sin (π/2) = 0 + i = i (relation 11) holds, then it follows that $-Et/\hbar = \pi/2$. From this, we can see that:

(23) $t = -\pi\hbar/2E$.

We will label this t as t_7. If $\hbar$ is defined as h/(2π) then:

(24) $t_7 = (-\pi h)/(4\pi E) = -h/4E$.

To finish our derivation, let us now go back to relation 8. There are two additional solutions for this relation, x_8 and x_9. Let $x_8 = -2\pi$ and $x_9 = 2\pi$.

From x_8 we can obtain t_8 very easily. If cos (-2π) + i sin (-2π) = 1 + 0 = 1 (relation 8) holds, then it follows immediately that $-Et/\hbar = -2\pi$. From this, we can see that:

(25) $t = (2\pi\hbar)/E$.

We will label this t as t_2. If $\hbar$ is defined as h/(2π) then:

(26) $t_8 = h/E$.

Similarly, from x_9 we can obtain t_9. If cos (2π) + i sin (2π) = 1 + 0 = 1 (relation 8) holds, then it follows that $-Et/\hbar = 2\pi$.

From this, we can see that:

(27) $t = (-2\pi\hbar)/E$.

We will label this t as t_9. If $\hbar$ is defined as $h/(2\pi)$ then:

(28) $t_9 = (2\pi nh)/(2\pi E) = (-h)/E$.

This concludes our derivation of our time operators. We will note that we have nine operators, the trivial operator, t_1 being zero. We have four operators that are positive, t_2, t_4, t_6, and t_8; and four that are negative, t_3, t_5, t_7, and t_9. We will only note before concluding that these operators are not applicable to all quantum mechanical systems, but only to those whose time-independent state functions do not depend on i, except perhaps for orientation in space. We would also like to note that for the ground state, E_0, the system can stay at this energy level continuously. Time is not quantized for <u>any</u> quantum-mechanical system at the ground level.

On the derivation of our time operators, we just have to note that t8 and t9, are incorporated for the sake of completeness. If we take a look at the unit circle, in figure 1, we will note that the traversal of the unit circle by these operators can be compared to the propagation of a wave, and that the operators coincide with the crests and throughs of a wave. Note that our operators make one complete traversal of the unit circle in each direction (supposedly as time goes forward, in a clockwise manner, or backwards, in a counterclockwise manner) until one complete wave is propagated, either forward in time or backwards in time.

Figure No. 1:

Time Operators Moving Along Unit Circle