GROUP PROPERTIES OF THE LORENTZ TRANSFORMATIONS

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Rigorosly applying Einstein's operational method of deducing the Lorentz transformation (LT), we get the vector LT and prove that non-collinear LT's form a group without requiring rotations of coordinate systems in this aim, as well. The result validates both the operational method and the LT, strengthening the relativity theory.

1. Introduction

Due to its change in length and direction with time, the radius vector of a geometrical point of an inertial („stationary" [1]) space with respect to the coordinate system (CS) of an observer „at rest"[1] (i.e., one not aimed by that motion) must -first of all- be traced physically, by one of the light signals emitted isotropically by a source attached to the origin of that CS. Then this radius vector can be drawn at a suitable scale and projected onto coordinate axes. By also projecting the radius vector of that point relative to a CS in the „stationary" space onto the coordinate axes of the last, are obtained relationships between its coordinates with respect to the two CS's, which depend on the travel time of that light signal. Moreover, whether an observer aimed by the motion of the „stationary" space traces by a light signal the radius vector of that point relative to his CS, a time depending relationship can be associated graphically to the former spatial relationships. Making this relationship to add light travel times as scalar quantities (an issue missed by Einstein, who chose only collinear light signals), we identified this set of new coordinate transformations with the standard LT in [1]. This result revealed both, the correctness of Einstein's 1905 derivation of the LT in [2] and the operational nature of the LT. Also tracing radius vectors by light signals, we deduce (in view of [1]) the vector LT in Sect. 2 of below, and the group properties of the LT's in Sect. 3. By proving that the non-collinear LT's form a group without requiring rotations of inertial CS's in this aim, we validate the LT, as well as the operational method used (unfortunately in an unexplicit manner) by Einstein in [2].

2. The Vector Lorentz Transformation

Consider the diagram in Fig. 1. The CS k moves rectilinearly with constant velocity v relative to the CS K along the direction $\mathbf{v}_0=\mathbf{v}/v$. To remove the dependence of OP and O'P on t* and O'Q/c, respectively, and so to get a time-relationship from a spacial-relationship divided by c, we define O'P' as time-axis and, like in [1], pass from Q and O' to Q_1 and O'_1 with $OP_1=\beta OP$ and $OO'_1=\beta OO'$. From the right triangles $O'_1Q_1P_1$ and OQP we have $\mathbf{r}'=\mathbf{Q_1P_1}+\mathbf{O'_1P_1}$ with $\mathbf{Q_1P_1}=\mathbf{r}-(\mathbf{r}.\mathbf{v}_0)\mathbf{v}_0$ and $\mathbf{O'_1P_1}=\mathbf{OP_1}-\mathbf{OO'_1}=\beta[(\mathbf{r}.\mathbf{v}_0)\mathbf{v}_0-\mathbf{v}T]$, that by noting $t'=(\mathbf{r}'.\mathbf{v}_0)/c$ and $T=(\mathbf{r}.\mathbf{v}_0)/c$, provides the vector LT as

$$\mathbf{r}'=\mathbf{r}-(\mathbf{r}.\mathbf{v}_0)\mathbf{v}_0+\beta[(\mathbf{r}.\mathbf{v}_0)\mathbf{v}_0-\mathbf{v}T],\ t'=\beta[T-(\mathbf{r}.\mathbf{v})/c^2]. \qquad (1)$$

From a diagram analogous to that in Fig. 1, describing the rectilinear motion of constant velocity $\mathbf{w}$ of a CS k relative to the CS K at absolute rest, we obtain analogously the vector LT

$$\mathbf{r}''=\mathbf{r}-(\mathbf{r}.\mathbf{w}_0)\mathbf{w}_0+\gamma[(\mathbf{r}.\mathbf{w}_0)\mathbf{w}_0-\mathbf{w}T],\ t''=\gamma[T-(\mathbf{r}.\mathbf{w})/c^2], \qquad (2)$$

where $\mathbf{w}_0=\mathbf{w}/w$, $\gamma=(1-w^2/c^2)^{-1/2}$ and $T=(\mathbf{r}.\mathbf{w}_0)/c$.

[1] For the relation between inertial ('stationary' [2]) CS's and CS's at absolute rest see [1].

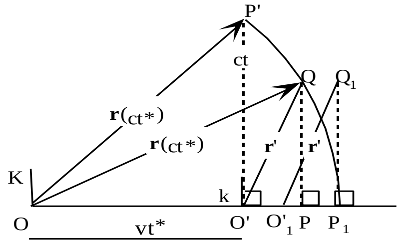

Figure 1

The CS k *and a light signal move, respectively, with constant velocity* **v** *and c relative to the origin* O *of* K, *reaching the points* Q *and* O*', respectively, at time* t*. O'P*' is the time*-*axis. By [1], we pass from* O', Q(**r**) *to* O', Q(β**r**), *having* OP_1=βOP, OO'_1=βOO', *and thus the vector LT (1).*

3. Group Properties

The main mathematical requirement for a set of coordinate transformations to form a group is that they to accomplish the transitivity property which stipulates that, successively performed, any two of them engender an equivalent one. The tracing of radius vectors by light signals is essential for proving operationally that collinear and non-collinear LT's form a group. Thus, by tracing $O'_{If}P_{IB}$ and $O'_{If}P_C$ in Figs. 2 and 3, respectively (O'_{If} in Figs. 2 and 3 being the origin of the CS K'_A at absolute rest associated to k_A in [3] by light signals, are obtained new LT's related to Eqs. (1) and (2), and similar to them. These signals will leave O'_{If} when O'_{If} and the origin of k_B in Fig. 2 (that of k'_B in Fig. 3) coincide, and will reach P_{IB} in Fig. 2 (P_{If}, P_C in Fig. 3) simultaneously with the light signal leaving O together with the origins of k_A and k_B, when the origin of k_B reaches O'_{IB} in Fig. 2 ($O'_{IB'}$, O'_{IB} in Fig. 3). As concerns the inverse transformation, it is associated with the motion with constant velocity -v of the origin of K from O' to O in Fig. 3 relative to the k now at absolute rest, and connects coordinates and times defining a different event. This because the CS Ξ associated to the moving K by $\xi=\beta^2x$ differs from that associated with the moving k in [4] by $\xi=\beta^2x'$.

3.1. *For Collinear LT's*

Consider the diagram in Fig. 2 for the collinear LT's (1), (2). At time t =0 the coinciding origins of k_A, k_B and a light signal leave the origin O of the CS K at absolute rest. The points O'_A, O'_B in Fig. 2 are reached by the origins of k_A, k_B, respectively, at time T, when the light signal reaches P(X). In accordance with [1], the LT's (1), (2) are written at the times βT and γT, respectively. The origin of k_A moves from O'_{IA} to O'_{If} in the time γT-βT. Analogously to the motion of k_B relative to K'_A in [3], we consider the motion of O'_{IB} relative to O'_{If}. We have from Fig. 2 $\mathbf{r}''=\mathbf{R}-\mathbf{O'}_{If}\mathbf{O'}_{IB}$ with $O'_{If}O'_{IB}=w\gamma T-v\gamma T$, $(\mathbf{R}.\mathbf{w}_0)=\gamma X-v\gamma T=\gamma(X-vT)=\gamma\beta^{-1}x'$, where x' is just ξ in [1], and

$x''=(\mathbf{r}''.\mathbf{w_0})=\gamma\beta^{-1}x'-(w-v)\gamma T=\gamma\beta^{-1}x'-\gamma(w-v)\beta t'-\gamma(w-v)\beta vx'/c^2=\gamma\beta^{-1}x'[1-(w-v)v/(c^2-v^2)]-\gamma\beta(w-v)t'=$

$\gamma\beta(1-wv/c^2)x'-\gamma\beta(w-v)t'$,

where t' is just τ in [1]. With **u** given by (7), $u_0=\mathbf{u}/u$, $\delta=(1-u^2/c^2)^{-1/2}$ and $\mathbf{v}_0$, $\mathbf{w}_0$, $\mathbf{u}_0$ all parallel, the relationships

$$\gamma\beta(1-wv/c^2)=\delta,\ \gamma\beta(w-v)=\delta u,\ (\mathbf{r}'.\mathbf{u}_0)=x' \qquad (3)$$

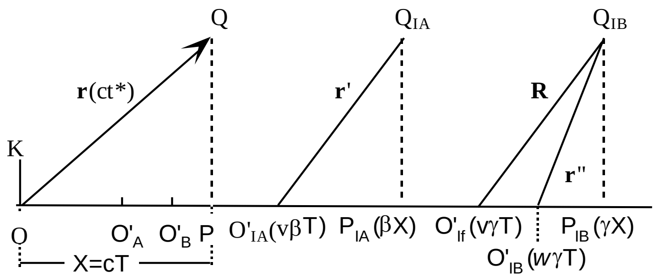

Figure 9

The origins of the CS's k_A, k_B *and a light signal move simultaneously with velocities v, w and c, respectively, along the common x',x'',x axis relative to* K. *By* [1], *we pass from* Q(**r**), O'_A, O'_B *to* $Q_{IA}(\beta\mathbf{r})$, O'_{IA} *and* $Q_{IB}(\gamma\mathbf{r})$, O'_{IB}, *obtaining the LT's* (1), (2), *respectively. The LT* (3) *is related to them by the motion of the origin of* k_A *from* O'_{IA} *to* O'_{If} *while the light signal travels from* P_{IA} *to* P_{IB}.

follow. From the right triangles $O'_{IB}Q_{IB}P_{IB}$ and $O'_{IA}Q_{IA}P_{IA}$ ($Q_{IA}P_{IA}=Q_{IB}P_{IB}$), we get the new vector LT

$$\mathbf{r}''=\mathbf{r}'-(\mathbf{r}'.\mathbf{u}_0)\mathbf{u}_0+\delta[(\mathbf{r}'.\mathbf{u}_0)\mathbf{u}_0-\mathbf{u}t'],\ t''=\delta[t'-(\mathbf{r}'.\mathbf{u})/c^2], \qquad (4)$$

where $t'=(\mathbf{r}'.\mathbf{u}_0)/c$ and $t''=(\mathbf{r}''.\mathbf{u}_0)/c$, which relates position vectors of geometrical points relative to k_B and k_A. Thus the transitivity condition is proved for collinear LT's.

3.2. For Non-Collinear LT's: Essential Test for Validity of LT

Consider the diagram in Fig. 3. At time t=0 the CS's k_A and k_B, whose origins coincide with that of CS K at absolute rest, start moving along non-parallel directions with constant velocities **v** and **w**, respectively. Light signals, tracing, respectively, the radius vectors OP_A, OP_B in Fig. 3, are required by the operational method developed in [1] to get the LT's (1), (2). To prove further that these non-collinear LT's form a group, the same method requires for a light signal and a CS k''_B, parallel to k_B, moving simultaneously at absolute velocities c and *w*-v along $O'_AO'_B$. Should result a new LT, in relation with (1) and (2). In this aim, the light signal and the origin of the CS k''_B must leave O'_A at time t=0, simultaneously with both, the CS k_B and an additional CS k'_B , also parallel to k_B, which covers in the time T a distance equal to $OO'_AO'_B$ along OP_A at a constant velocity *w*. Therefore, we have

$$|\mathbf{w}-\mathbf{v}|T=(w-v)T.$$

Moreover, by reporting the motion of k_B to the CS K'_A at absolute rest -associated to k_A in [3]-, we have

$$(\boldsymbol{w}-\mathbf{v})T=(T-wvT/c^2)u\mathbf{u}_0,$$

with $u=(w-v)/(1-wv/c^2)$, $\mathbf{u}_0=(\mathbf{w}-\mathbf{v})/|\mathbf{w}-\mathbf{v}|$, and so the equation

$$\mathbf{w}-\mathbf{v}=(w-v)\mathbf{u}_0, \qquad (5)$$

of whole operational nature. Eq. (5) stands for the standard relativistic law for the composition of non-parallel velocities -recall that the relativistic law for the composition of parallel velocities was also obtained in [3] as an operational law. As concerns the standard relativistic law for the composition of non-parallel velocities, it was obtained with the ignorance of the fact that any uniform translatory motion is (as shown in [3]) referred to a CS at absolute rest, and predicted the famous Thomas precession experimentally unproved [6].

At the times βT, γT the light signals leaving O simultaneously with k_A, k_B and k'_B reach, respectively, P_{IA} and P_{If}, P_{IB}, while the origins of k_A and k_B arrive, respectively, at O'_{IA} and O'_{If}, O'_{IB}. In accordance with [3], O'_{If} is the origin of the CS K'_A at absolute rest at time γT. By the above definition of k'_B and k''_B, the origin of k'_B finds at time γT at a distance equal to $O'_{If}O'_{IB}$ from O'_{If} along OP_{If}, namely at $O'_{IB'}$ in Fig. 3. The light signals leaving O'_{If} simultaneously with the origins of k'_B and k''_B will travel equal distances along the directions of motion of k'_B and k''_B, viz. $O'_{If}P_{If}=O'_{If}P_C$ Since $O'_{If}P_{If}$ is the projection of R onto the direction of v, $O'_{If}P_C$ will be the projection of a vector $\mathbf{R}'$ of magnitude R that makes with $\mathbf{u}_0$ an angle equal to that $\mathbf{R}$ makes with $\mathbf{v}$. From $O'_{If}P_{If}=(\mathbf{R}.\mathbf{v}_0)=\gamma[(\mathbf{r}.\mathbf{v}_0)-vT]$ and the equation $(\mathbf{r}'.\mathbf{v}_0)=\beta[(\mathbf{r}.\mathbf{v}_0)-vT]$, resulting from the first of Eqs. (1), we have $(\mathbf{R}.\mathbf{v}_0)=\gamma\beta^{-1}(\mathbf{r}'.\mathbf{v}_0)$ with

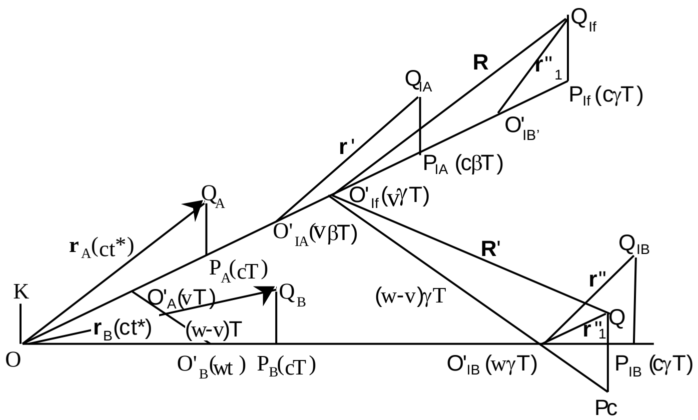

Figure 3

By [1], *the origins of the CS's* k_A, k_B *moves with constant velocities* **v** *and* **w**, *respectively, simultaneously with two light signals, reaching at time T the points* O'_A, O'_B *and* P_A, P_B, *respectively. Also by* [1], *we pass further from these points to* O'_{IA}, O'_{IB} *and* P_{IA}, P_{IB},*respectively, writting the LT's* (1), (2). *The additional motion of the origin of* k_A *and the origin of the suitable light signal from* O'_{IA}, P_{IA} *to* O'_{If}, P_{If}, *until the other light signal reaches* P_{IB}, *enables us to deduce the related LT* (22).

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$$(\mathbf{R}.\mathbf{v}_0)\mathbf{u}_0=\gamma\beta^{-1}(\mathbf{r}'.\mathbf{v}_0)\mathbf{u}_0. \qquad (6)$$

By inserting (6), the inverse of the last of Eqs. (1), and Eq. (5) in $(\mathbf{R}.\mathbf{v}_0)\mathbf{u}_0\text{-}\mathbf{u}'\gamma T$, we obtain

$$(\mathbf{R}.\mathbf{v}_0)\mathbf{u}_0\text{-}(\mathbf{w}\text{-}\mathbf{v})\gamma T=\gamma\beta^{-1}(\mathbf{r}'.\mathbf{v}_0)\mathbf{u}_0\text{-}\mathbf{u}_0(w\text{-v})\gamma\beta t'\text{-}\mathbf{u}_0(w\text{-v})\gamma\beta v(\mathbf{r}'.\mathbf{v}_0)/c^2=\gamma\beta^{-1}(\mathbf{r}'.\mathbf{v}_0)\mathbf{u}_0[1\text{-}(w\text{-v})v/(c^2\text{-}v^2)]\text{-}\mathbf{u}_0(w\text{-v})\gamma\beta t'=\gamma\beta(1\text{-}wv/c^2)(\mathbf{r}'.\mathbf{v}_0)\mathbf{u}_0\text{-}\mathbf{u}_0(w\text{-v})\gamma\beta t'.$$

In view of Eqs. (4), also valid for w, we have

$$\mathbf{O}'_{IB}\mathbf{P}_C=(\mathbf{R}.\mathbf{v}_0)\mathbf{u}_0\text{-}(\mathbf{w}\text{-}\mathbf{v})\gamma T=\delta[(\mathbf{r}'.\mathbf{v}_0)\text{-}ut']\mathbf{u}_0=\delta[(\mathbf{r}'.\mathbf{u}_0)\mathbf{u}_0\text{-}\mathbf{u}t'].$$

Because $Q_{If}P_{If}=Q_{IB}P_{IB}=QP_C$ by virtue of $Q_AP_A=Q_BP_B$, and $|\mathbf{r}''_1| = |\mathbf{O}'_{IB'}\mathbf{Q}_{If}| = |\mathbf{O}'_{IB}\mathbf{Q}|$ with $\mathbf{O}'_{IB}\mathbf{Q}=\mathbf{Q}\mathbf{P}_C+\mathbf{O}'_{IB}\mathbf{P}_C$, we have, respectively, $\mathbf{Q}\mathbf{P}_C=\mathbf{r}'\text{-}(\mathbf{r}'.\mathbf{u}_0)\mathbf{u}_0$ and

$$\mathbf{r}''_1=\mathbf{r}'\text{-}(\mathbf{r}'.\mathbf{u}_0)\mathbf{u}_0+\delta[(\mathbf{r}'.\mathbf{u}_0)\mathbf{u}_0\text{-}\mathbf{u}t'],\ t''=\delta[t'\text{-}(\mathbf{r}'.\mathbf{u})/c^2], \qquad (7)$$

where $t''=(\mathbf{r}''_1.\mathbf{u}_0)/c=(\mathbf{r}''_1.\mathbf{w}_0)/c$. Similar to the collinear LT's (1) and (2), the resulting vector LT (7) proves that the non-collinear LT's satisfy the transitivity property, forming a group without requiring rotations of 'stationary' (inertial) CS's in this aim. This result validates the LT.

4. Conclusions

Deducing the vector LT, as well as that non-collinear LT's form a group without requiring rotations of inertial CS's in this aim, by an operational method, we validate the LT, this method, implicitly Einstein's 1905 derivation of the LT, what strengthens substancially the special relativity and the area of applications of the relativistic physics.

References

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