Relativistic exact expressions for wave refraction in a generally moving medium. Application to the gravitational lense effect

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Abstract

The law for the refraction of a wave when the two media and the interface are moving with relativistic velocities is given in an exact form, thus improving the approximate expression given in a previous paper [G. Cavalleri and E. Tonni, Phys. Rev. E **57**, 3478 (1998)]. The treatment is then extended to a generally moving medium with variable refractive index in both the nonrelativistic and relativistic case. The nonrelativistic extension, put in a convenient form for computer's use, represents an improvement of an another previous paper [A. Ascoli, C. Bernasconi, and G. Cavalleri, Phys. Rev. E **54**, 6291 (1996)]. The treatment is then applied to the light emitted by a distant quasar and refracted by the relativistically moving gas of an intermediate star or galaxy. The corresponding deviation of light is additional to that produced by the gravitational lense effect. Using standard profiles of density the correction turns out to be negligible if only the average galactic gas is considered. A possible sizeable contribution can arise if one of the two light rays passes close to a star but the probability of such an event turns out to be small. This exact check of a possible contribution due to relativistic refraction makes it certain that the lense effect is due to gravity, with the rare exception that the light traversing a hedge-on galaxy passes close to a star.

I. INTRODUCTION

The refraction of a wave beam on an interface moving with velocity $\mathbf{V}$ and separating two media with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, with both $|\mathbf{u}_1|$ and $|\mathbf{u}_2|$ small compared with the speed c of light, was given in Ref. [1] in an exact way. In Ref. [2] the relativistic version was given, although with the introduction of a spurious term of first order in u_2/c, erroneously considered as an improvement even for the nonrelativistic version. It is instructive to see how this (first order) error arose. The method used in Ref. [2] is illustrated in Fig. 1. The error is due to the fact that a wave front (perpendicular to the wave rays) is correctly obtained by the Huygens construction only in the reference system S_0 at rest with the fluid. This is what done in step 2 of Ref. [1], which was therefore correct. Consider the case of $u_{02} = 0$ for S_0, find the direction of the refracted ray and the corresponding wave front. Then consider another observer S for which $\mathbf{u}_{02}$ is along the direction of the refracted ray for S_0. It is obvious that the direction does not change, whereas, with the construction of Fig. 1, both this direction and the one of the wave front vary depending on $|\mathbf{u}_{02}|$.

In Sec. II we summarize the nonrelativistic exact result of Ref. [1] and then we extend them by giving an explicit expression to be used in a computer to obtain the trajectory of a wave beam in a regularly moving medium with a wave speed $|\mathbf{c}_0(\mathbf{r})|$ assigned as a function of position $\mathbf{r}$.

In Sec. III we summarize the relativistic results of Ref. [2], making them exact by dropping the spurious additional term. Then we extend the exact results to a regularly moving fluid.

The relativistic results are applied in Sec. IV to the light emitted by a distant quasar and refracted by the moving gas of an intermediate either star or galaxy. To calculate the deviation of the light we must know the refraction index n_r that is evaluated in Sec. IV A. Then, with the use of the found n_r, we estimate in Sec. IV B the relative importance of deviation due to refraction with respect to the one due to gravitation.

We conclude in Sec. V commenting our result.

II. EXTENSION OF THE EXACT, NONRELATIVISTIC REFRACTION TO A REGULARLY MOVING MEDIUM

Before extending the nonrelativistic treatment to a regularly moving fluid, we first summarize the results of Ref. [1].

A. First step

If $\mathbf{u}_1$ is the velocity of the first fluid (through which the incoming wave is propagating before refraction) with respect to the laboratory system S, we have to consider the wave in the system S_0 at rest with fluid 1. Let $\mathbf{c}_1$ and $\mathbf{c}_{01}$ be the wave velocities in S and S_0, respectively, related to each other by

$$\mathbf{c}_1 = \mathbf{c}_{01} + \mathbf{u}_1 \ . \tag{1}$$

Take the unit vector $\hat{\mathbf{n}}$ perpendicular to the mobile interface and directed so as $\mathbf{c}_{01} \cdot \hat{\mathbf{n}} > 0$. Then the incident angles θ_{01} and θ_1 in S_0 and S, respectively, are given by

$$\cos\theta_{01} = \mathbf{c}_{01} \cdot \hat{\mathbf{n}}/c_{01} \tag{2}$$

and

$$\cos\theta_1 = \mathbf{c}_1 \cdot \hat{\mathbf{n}}/c_1 = (c_{01}\cos\theta_{01} + \mathbf{u}_1 \cdot \hat{\mathbf{n}})/c_1 \ . \tag{3}$$

Notice that c_{01} is the speed of the wave in the fluid at rest and it is therefore a known quantity.

B. Second step

If $\mathbf{V}$ is the local velocity of the interface σ in the laboratory system S, its velocity $\mathbf{V}_0$ in the reference system S_0 at rest with the first fluid is

$$\mathbf{V}_0 = \mathbf{V} - \mathbf{u}_1 \ . \tag{4}$$

What is effective is the component

$$V_{0\perp} = \mathbf{V}_0 \cdot \hat{\mathbf{n}} \tag{5}$$

of $\mathbf{V}_0$ along the normal to the interface.

The unit vector $\hat{\mathbf{n}}$ is drawn so that $\mathbf{c}_{01} \cdot \hat{\mathbf{n}} > 0$. Media 1 and 2 contain the incident and refracted wave, respectively. If σ were at rest, there would be no ambiguity about which one is the incident wave. However, if $V_{0\perp} > c_{01}\cos\theta_{01}$, it is the interface σ that reaches the fleeing wave and we have to exchange medium 1 for 2. Consequently, if σ is at rest, medium 1 is always the one <u>not</u> containing $\hat{\mathbf{n}}$ (drawn starting from the interface). If σ is in motion, medium 1 is the one not containing $\hat{\mathbf{n}}$ only if

$$s = \mathrm{sgn}[(\mathbf{c}_{01} - \mathbf{V}_0) \cdot \hat{\mathbf{n}}] \tag{6}$$

is plus, medium 1 is that containing $\hat{\mathbf{n}}$ if s is minus.

Using the Huygens construction, the cosine of the refracted angle θ_{02} (measured in the system S_0 at rest with fluid 1) turns out to be given by Eq. (15) of Ref. [1]

$$\cos\theta_{02} = \frac{m\,q + s\,p\,(m^2 + p^2 - q^2)^{1/2}}{m^2 + p^2} \ , \tag{7}$$

where s is given by Eq. (6) and

$$m = (\mathbf{V} - \mathbf{u}_1) \cdot \hat{\mathbf{n}} \left\{ 1 - \left[\frac{(\mathbf{c}_1 - \mathbf{u}_1) \cdot \hat{\mathbf{n}}}{|\mathbf{c}_1 - \mathbf{u}_1|} \right]^2 \right\}^{1/2} \ , \tag{8}$$

$$p = c_{01} - (\mathbf{V} - \mathbf{u}_1) \cdot \hat{\mathbf{n}} \, \frac{(\mathbf{c}_1 - \mathbf{u}_1) \cdot \hat{\mathbf{n}}}{|\mathbf{c}_1 - \mathbf{u}_1|} \ , \tag{9}$$

$$q = c_{22} \left\{ 1 - \left[\frac{(\mathbf{c}_1 - \mathbf{u}_1) \cdot \hat{\mathbf{n}}}{|\mathbf{c}_1 - \mathbf{u}_1|} \right]^2 \right\}^{1/2} \ . \tag{10}$$

C. Third step

The velocity $\mathbf{c}_{02}$ (measured in the system S_0 at rest with fluid 1) in medium 2 assumed at rest with fluid 1 is,

$$\mathbf{c}_{02} = c_{22} \left(\hat{\mathbf{n}} \cos\theta_{02} + \hat{\boldsymbol{\sigma}} \sin\theta_{02} \right) \ , \tag{11}$$

where c_{22} is the speed of the wave in medium 2 if at rest, θ_{02} is given by Eq. (7) and

$$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{b}} \times \hat{\mathbf{n}} = (\hat{\mathbf{n}} \times \hat{\mathbf{c}}_{01} / \sin\theta_{01}) \times \hat{\mathbf{n}} = (\hat{\mathbf{c}}_{01} - \hat{\mathbf{n}} \cos\theta_{01}) / \sin\theta_{01} \ . \tag{12}$$

The velocity $\mathbf{c}^*_{02}$, still measured in S_0 (at rest with fluid 1), but in fluid 2 in motion is

$$\mathbf{c}^*_{02} = \mathbf{c}_{02} + \mathbf{u}_{02} = \mathbf{c}_{02} + \mathbf{u}_2 - \mathbf{u}_1 \ . \tag{13}$$

Finally, the velocity $\mathbf{c}_2$ measured in the laboratory system S is

$$\mathbf{c}_2 = \mathbf{c}^*_{02} + \mathbf{u}_1 = \mathbf{c}_{02} + \mathbf{u}_2 \tag{14}$$

and

$$\cos\theta_2 = \hat{\mathbf{c}}_2 \cdot \hat{\mathbf{n}} \ . \tag{15}$$

D. Application to inhomogeneous, regularly moving media

Let us apply the above results to a regularly moving inhomogeneous medium, for example a compressible fluid. To obtain the trajectory of a wave ray by means of a computer, we consider the medium as a succession of thin adjacent layers so that the speed $|\mathbf{c}_0(\mathbf{r})| = c_{22}(\mathbf{r})$ of the wave with respect to the local medium is independent of the considered small volume having velocity $\mathbf{u}(\mathbf{r})$. Let $\mathbf{c}(\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \mathbf{c}_0(\mathbf{r})$ be the initial, known velocity of a local wave ray, where the dependence on the position $\mathbf{r}$ substitutes the subscript 1 of Eq. (1). Since the velocity field $\mathbf{u}(\mathbf{r})$ of the medium must be assigned, we must know the initial velocity $\mathbf{c}_0(\mathbf{r})$ with respect to the local medium. We now give a displacement $\delta\mathbf{r}^* = \mathbf{c}(\mathbf{r})\,\delta t$ where we can fix either $|\delta\mathbf{r}^*|$ or δt. In the new position we know both $\mathbf{u}(\mathbf{r} + \mathbf{c}\,\delta t)$ and $|\mathbf{c}_0(\mathbf{r} + \mathbf{c}\,\delta t)|$, while the direction $\hat{\mathbf{c}}_0(\mathbf{r} + \mathbf{c}\,\delta t)$ is derived from the law of refraction. We then give the effective displacement

$$\delta\mathbf{r} = \frac{\delta t}{2} \left\{ \mathbf{c}(\mathbf{r}) + \mathbf{c}[\mathbf{r} + \mathbf{c}(\mathbf{r})\,\delta t] \right\} \ , \tag{16}$$

thus obtaining a polygonal path tangent to the true curve corresponding to $\delta t = 0$.

We derive from Eq. (14)

$$\mathbf{c}[\mathbf{r} + \mathbf{c}(\mathbf{r})\,\delta t] = \mathbf{u}(\mathbf{r} + \mathbf{c}\,\delta t) + \mathbf{c}_0(\mathbf{r} + \mathbf{c}\,\delta t) \ . \tag{17}$$

where $\mathbf{c}(\mathbf{r} + \mathbf{c}\,\delta t)$ and $\mathbf{u}(\mathbf{r} + \mathbf{c}\,\delta t)$ stand for $\mathbf{c}_2$ and $\mathbf{u}_2$, respectively, while

$$\mathbf{c}_0(\mathbf{r} + \mathbf{c}\,\delta t) = |\mathbf{c}_0(\mathbf{r} + \mathbf{c}\,\delta t)| \left(\hat{\mathbf{n}}\, \cos\theta_{02} + \hat{\boldsymbol{\sigma}}\, \sin\theta_{02} \right) \ , \tag{18}$$

$|\mathbf{c}_0(\mathbf{r} + \mathbf{c}\,\delta t)|$ standing for c_{22} of Eq. (11). The unit vector $\hat{\mathbf{n}}$ perpendicular to the interface, between two adjacent layers of the medium (a compressible fluid) can be derived from the local characteristic of the medium, for instance its density, or the corresponding absolute value $|\mathbf{c}_0(\mathbf{r})|$. The gradient of the latter gives the direction of $\hat{\mathbf{n}}$. For its sign, we remind that we have chosen $\hat{\mathbf{n}}$ such that $\hat{\mathbf{n}} \cdot \mathbf{c}_0(\mathbf{r}) > 0$. Consequently, we can write

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}^* \, \frac{\mathbf{c}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}^*}{|\mathbf{c}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}^*|} \qquad \text{with} \qquad \hat{\mathbf{n}}^* = \frac{\boldsymbol{\nabla} \left| \mathbf{c}_0 \left\{ \mathbf{r} + \frac{1}{2}\, \mathbf{c}(\mathbf{r})\, \delta t \right\} \right|}{\left| \boldsymbol{\nabla} \left| \mathbf{c}_0 \left\{ \mathbf{r} + \frac{1}{2}\, \mathbf{c}(\mathbf{r})\, \delta t \right\} \right| \right|} \ . \tag{19}$$

Then, the unit vector $\hat{\boldsymbol{\sigma}}$, whose direction is given by intersection of the interface with the incidence plane, can be derived from Eq. (12) and, in the present notations of a generally inhomogeneous medium, reads

$$\hat{\boldsymbol{\sigma}}(\mathbf{r}) = \hat{\mathbf{n}}(\mathbf{r}) \times [\mathbf{c}_0(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r})]\, |\mathbf{c}_0(\mathbf{r}) \times \hat{\mathbf{n}}(\mathbf{r})|^{-1} \ . \tag{20}$$

Substituting Eqs. (17) and (18) into Eq. (16) we obtain for the elementary step $\delta\mathbf{r}$

$$\delta\mathbf{r} = \frac{\delta t}{2}\left\{\mathbf{u}(\mathbf{r}) + \mathbf{u}\left[\mathbf{r} + \delta t\left(\mathbf{u}(\mathbf{r}) + \mathbf{c}_0(\mathbf{r})\right)\right] + \mathbf{c}_0(\mathbf{r}) + \left|\mathbf{c}_0\left[\mathbf{r} + \delta t\left(\mathbf{u}(\mathbf{r}) + \mathbf{c}_0(\mathbf{r})\right)\right]\right|\left(\hat{\mathbf{n}}\cos\theta_{02} + \hat{\boldsymbol{\sigma}}\sin\theta_{02}\right)\right\}\ , \tag{21}$$

where the velocity field $\mathbf{u}(\mathbf{r})$ of the compressible fluid is known, the initial value of $\mathbf{c}_0(\mathbf{r})$ is assigned (and in a subsequent step is the resulting value given by Eq. (17) of the preceding step), $\hat{\mathbf{n}}$ and $\hat{\boldsymbol{\sigma}}$ are given by Eqs. (19) and (20), respectively, $\cos\theta_{02}$ by Eq. (7) where the coefficients m, p, and q contained in the expression of $\cos\theta_{02}$ are translated for the continuum from Eqs. (8)-(10) and read

$$m = \left\{\mathbf{V}\left[\mathbf{r} + \frac{1}{2}\mathbf{c}(\mathbf{r})\,\delta t\right] - \mathbf{u}(\mathbf{r})\right\}\cdot\hat{\mathbf{n}}\left\{1 - \left[\frac{\mathbf{c}_0(\mathbf{r})\cdot\hat{\mathbf{n}}}{|\mathbf{c}_0(\mathbf{r})|}\right]^2\right\}^{1/2}\ , \tag{22}$$

$$p = |\mathbf{c}_0(\mathbf{r})| - \left\{\mathbf{V}\left[\mathbf{r} + \frac{1}{2}\mathbf{c}(\mathbf{r})\,\delta t\right] - \mathbf{u}(\mathbf{r})\right\}\cdot\hat{\mathbf{n}}\,\frac{\mathbf{c}_0(\mathbf{r})\cdot\hat{\mathbf{n}}}{|\mathbf{c}_0(\mathbf{r})|}\ , \tag{23}$$

$$q = c_0\left[\mathbf{r} + \delta t\left(\mathbf{u}(\mathbf{r}) + \mathbf{c}_0(\mathbf{r})\right)\right]\left\{1 - \left[\frac{\mathbf{c}_0(\mathbf{r})\cdot\hat{\mathbf{n}}}{|\mathbf{c}_0(\mathbf{r})|}\right]^2\right\}^{1/2}\ , \tag{24}$$

where $\mathbf{V}\left[\mathbf{r} + \mathbf{c}(\mathbf{r})\,\delta t/2\right]$ is the velocity of the interface (which must be assigned).

To obtain the following step we repeat the same procedure. In particular, we subtract $\mathbf{u}(\mathbf{r})$ from the true velocity $\mathbf{c}(\mathbf{r})$ to obtain the velocity $\mathbf{c}_0(\mathbf{r})$ of the wave ray in the system at rest with the local fluid. Then we apply the law of refraction and add the relative velocity $\mathbf{u}(\mathbf{r} + \delta\mathbf{r}) - \mathbf{u}(\mathbf{r})$ of the next layer with respect to the preceding. To obtain the velocity in the laboratory system S we then add $\mathbf{u}(\mathbf{r})$ that therefore cancel $-\mathbf{u}(\mathbf{r})$ thus getting Eq. (17) (applied to the next step).

One can therefore obtain the trajectory of a wave beam by computer. The elementary step $\delta\mathbf{r}$ is given by Eq. (21) with $\cos\theta_{02}$ (hence $\sin\theta_{02}$) given by Eq. (7), where the coefficients m, p, and q are expressed by Eqs. (22)-(24). For the first step, $\mathbf{u}(\mathbf{r})$ and $\mathbf{c}_0(\mathbf{r})$ must be assigned as initial conditions. Once calculated the first $\delta\mathbf{r}$ by Eq. (16) or, in a more detailed way, by Eq. (21), both $\mathbf{u}(\mathbf{r}+\delta\mathbf{r})$ and $|\mathbf{c}_0(\mathbf{r}+\delta\mathbf{r})|$ are known since the velocity field $\mathbf{u}(\mathbf{r})$ of the medium (for instance a fluid) is assigned and $|\mathbf{c}_0(\mathbf{r})|$ is known as a property of the medium, for example by assigning the refractive index $n(\mathbf{r})$. Then $\mathbf{u}(\mathbf{r}+\delta\mathbf{r})$ and $|\mathbf{c}_0(\mathbf{r}+\delta\mathbf{r})|$ become the initial known values of the subsequent step. Finally, step by step, the computer gives the whole trajectory that can easily be visualized.

In a future paper, the results of this section will be applied to the filament model [3].

III. DERIVATION OF THE EXPRESSION OF RELATIVISTIC REFRACTION

We first summarize the results of Ref. [2], deprived of the additional, second term which, actually, gave an error of second order, so as to obtain the exact relativistic version of the exact nonrelativistic treatment.

A. First step

We choose a system S of Cartesian axes with the x axis parallel to the velocity $\mathbf{u}_1$ of medium 1 (through which the incoming wave is propagating before refraction). We have to consider the wave in the system S_0 at rest with fluid 1. Let $\mathbf{c}_1$ be the wave velocity in S and $\hat{\mathbf{n}}$ be the unit vector perpendicular to the mobile interface σ and directed from medium 1 to medium 2. As written in Ref. [2], $\hat{\mathbf{n}}$ is not the transformed unit vector of $\hat{\mathbf{n}}_\sigma$ perpendicular to the interface σ in the system S_σ at rest with σ, but the unit vector perpendicular to σ as seen by S; moreover, the unit vector $\hat{\mathbf{n}}_0$ perpendicular to σ as seen by S_0 is not equal to $\hat{\mathbf{n}}$, but it is given by Eq. (21) of Ref. [2]. In any case, both $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_0$ can be derived from Eq. (19) calculated in S and S_0, respectively.

The wave velocity $\mathbf{c}_{01}$ in S_0 is given, since $\mathbf{u}_1 = u_1\,\hat{\mathbf{e}}_x$, by

$$\mathbf{c}_{01} = (1 - \mathbf{u}_1\cdot\mathbf{c}_1/c^2)^{-1}\left[(c_{1x} - u_1)\,\hat{\mathbf{e}}_x + \gamma_1^{-1}\left(c_{1y}\,\hat{\mathbf{e}}_y + c_{1z}\,\hat{\mathbf{e}}_z\right)\right]\ , \tag{25}$$

where

$$\gamma_1 = (1 - u_1^2/c^2)^{-1/2} \tag{26}$$

is the usual relativistic factor and c is the speed of light in vacuum.

Then, we take the unit vector $\hat{\mathbf{n}}_0$ directed so as $\mathbf{c}_{01} \cdot \hat{\mathbf{n}}_0 > 0$.

The incident angles θ_{01} and θ_1 in S_0 and S, respectively, are given by

$$\cos\theta_{01} = \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{c}}_{01} \quad \text{and} \quad \cos\theta_1 = \hat{\mathbf{n}} \cdot \hat{\mathbf{c}}_1 \ . \tag{27}$$

B. Second step

In the relativistic case, the local velocity $\mathbf{V}_0$ of the interface in the system S_0 is given by

$$\mathbf{V}_0 = (1 - \mathbf{u}_1 \cdot \mathbf{V}/c^2)^{-1} \left[(V_x - u_1)\,\hat{\mathbf{e}}_x + \gamma_1^{-1}(V_y\,\hat{\mathbf{e}}_y + V_z\,\hat{\mathbf{e}}_z)\right] \ , \tag{28}$$

with γ_1^{-1} given by Eq. (26).

After the Huygens construction, the cosine of the refracted angle θ_{02} (measured in the system S_0) turns out to be still given by Eq. (7), where s is expressed by

$$s = \mathrm{sgn}[(\mathbf{c}_{01} - \mathbf{V}_0) \cdot \hat{\mathbf{n}}_0] \ , \tag{29}$$

which differs from Eq. (6) since $\hat{\mathbf{n}}_0 \neq \hat{\mathbf{n}}$ and $\mathbf{V}_0$ is now given by Eq. (28) instead of Eq. (4). Moreover, m, p, and q are given by

$$m = \mathbf{V}_0 \cdot \hat{\mathbf{n}}_0 \, \sin\theta_{01} \ , \quad p = c_{01} - \mathbf{V}_0 \cdot \hat{\mathbf{n}}_0 \, \cos\theta_{01} \ , \quad q = c_{02} \, \sin\theta_{01} \ . \tag{30}$$

C. Third step

The velocity $\mathbf{c}_{02}$ (measured in the system S_0 at rest with fluid 1) in medium 2 assumed at rest with fluid 1 is still given by Eq. (11), where instead of $\hat{\mathbf{n}}$ we write $\hat{\mathbf{n}}_0$ and instead of $\hat{\boldsymbol{\sigma}}$ we put $\hat{\boldsymbol{\sigma}}_0$, expressed by

$$\hat{\boldsymbol{\sigma}}_0 = \hat{\mathbf{b}}_0 \times \hat{\mathbf{n}}_0 = (\hat{\mathbf{n}}_0 \times \hat{\mathbf{c}}_{01}/\sin\theta_{01}) \times \hat{\mathbf{n}}_0 = (\hat{\mathbf{c}}_{01} - \hat{\mathbf{n}}_0 \cos\theta_{01})/\sin\theta_{01} \ . \tag{31}$$

The velocity $\mathbf{u}_{02}$ of medium 2 with respect to S_0 is given by

$$\mathbf{u}_{02} = (1 - \mathbf{u}_1 \cdot \mathbf{u}_2/c^2)^{-1} \left[(u_{2x} - u_1)\,\hat{\mathbf{e}}_x + \gamma_1^{-1}(u_{2y}\,\hat{\mathbf{e}}_y + u_{2z}\,\hat{\mathbf{e}}_z)\right] \ , \tag{32}$$

where γ_1 is given by Eq. (26).

In order to obtain the velocity $\mathbf{c}_{02}^*$ of the refracted wave as measured by S_0, we must transform $\mathbf{c}_{02}$ measured in the reference system S_2 (at rest with fluid 2 and therefore moving with velocity $\mathbf{u}_{02}$ with respect to S_0) to $\mathbf{c}'_{02}$ measured in the system S_0 (at rest with fluid 1). The expression of $\mathbf{c}'_{02}$, calculated in Ref. [2], is here reported

$$\mathbf{c}_{02}^* = \mathbf{c}'_{02} = \frac{d\mathbf{r}_0}{dt_0} = \gamma_{02}^{-1} \, \frac{\mathbf{c}_{02} + \hat{\mathbf{u}}_{02}\,(\gamma_{02} - 1)\,\hat{\mathbf{u}}_{02} \cdot \mathbf{c}_{02} + \gamma_{02}\,\mathbf{u}_{02}}{1 + \mathbf{u}_{02} \cdot \mathbf{c}_{02}/c^2} \ , \tag{33}$$

where

$$\gamma_{02} = (1 - u_{02}^2/c^2)^{-1/2} \ . \tag{34}$$

For u_{02}, $c_{02} \ll c$, we reduce to the nonrelativistic case. For $u_{02} \ll c$ but c_{02} relativistic, expanding Eq. (33) to first order gives

$$\mathbf{c}_{02}^* \simeq \mathbf{c}_{02} + \mathbf{u}_{02} - \mathbf{c}_{02}\,\mathbf{u}_{02} \cdot \mathbf{c}_{02}/c^2 \ . \tag{35}$$

To obtain $\mathbf{c}_2$ we transform $\mathbf{c}_{02}^*$ from the system S_0 (at rest with medium 1) to the laboratory system S,

$$\mathbf{c}_2 = (1 + \mathbf{u}_1 \cdot \mathbf{c}_{02}^*/c^2)^{-1} \left[(c_{02x}^* + u_1)\,\hat{\mathbf{e}}_x + \gamma_1^{-1}\,(c_{02y}^*\,\hat{\mathbf{e}}_y + c_{02z}^*\,\hat{\mathbf{e}}_z)\right] \ , \tag{36}$$

where $\mathbf{u}_1$ is the known velocity of fluid 1 with respect to the laboratory observer, $\mathbf{c}_{02}^*$ is given by Eq. (33).

Finally,

$$\cos\theta_2 = \hat{\mathbf{n}} \cdot \mathbf{c}_2/c_2 \ , \tag{37}$$

where $\mathbf{c}_2$ is given by Eq. (36) and $\hat{\mathbf{n}}$ is the local normal to the interface as seen by the laboratory observer S.

D. Application to inhomogeneous, generally moving media

Let us apply the above results to a regularly moving inhomogeneous medium, for example a compressible fluid. We proceed in a similar way to the nonrelativistic case. Since the initial velocity of a local wave ray $\mathbf{c}(\mathbf{r})$ is known and the velocity field $\mathbf{u}(\mathbf{r})$ of the medium must be assigned, we can know the initial velocity $\mathbf{c}_0(\mathbf{r})$ with respect to the local medium by Eq. (25), where the dependence on the position $\mathbf{r}$ substitutes the subscript 1. We now give a displacement $\delta\mathbf{r}^* = \mathbf{c}(\mathbf{r})\,\delta t$ where we can fix either $|\delta\mathbf{r}^*|$ or δt. In the new position we know both $\mathbf{u}(\mathbf{r}+\mathbf{c}\,\delta t)$ and $|\mathbf{c}_0(\mathbf{r}+\mathbf{c}\,\delta t)|$, while the direction $\hat{\mathbf{c}}_0(\mathbf{r}+\mathbf{c}\,\delta t)$ is derived from the law of refraction. Precisely, it is

$$\mathbf{c}_0(\mathbf{r}+\mathbf{c}\,\delta t) = |\mathbf{c}_0(\mathbf{r}+\mathbf{c}\,\delta t)|\,(\hat{\mathbf{n}}_0\,\cos\theta_{02} + \hat{\boldsymbol{\sigma}}_0\,\sin\theta_{02})\ , \tag{38}$$

where θ_{02} is given by Eq. (7) while $\hat{\mathbf{n}}_0$ and $\hat{\boldsymbol{\sigma}}_0$ are calculated by means of Eqs. (19) and (20), respectively, calculated in the system S_0 at rest with the local fluid. In nonrelativistic physics it is $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0$, $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}_0$, whereas these equalities are no longer valid in relativity as discussed in Sec. IIA of Ref. [2]. The unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_0$ perpendicular to the interface, between two adjacent layers of the medium (a compressible fluid) can be derived from the local characteristic of the medium, for instance its density, or the corresponding absolute value $|\mathbf{c}(\mathbf{r})|$ and $|\mathbf{c}_0(\mathbf{r})|$, respectively. The gradients of the latters gives the direction of the normals. For its sign, we remind that we have chosen $\hat{\mathbf{n}}_0$ such that $\hat{\mathbf{n}}_0\cdot\mathbf{c}_0(\mathbf{r}) > 0$. Consequently, from Eq. (19) we can obtain both $\hat{\mathbf{n}}_0$ [putting in Eq. (19) $\hat{\mathbf{n}}_0$ instead of $\hat{\mathbf{n}}$] and $\hat{\mathbf{n}}$ [putting in Eq. (19) $\mathbf{c}(\mathbf{r})$ instead of $\mathbf{c}_0(\mathbf{r})$]. The coefficients m, p, and q appearing in Eq. (7) are given by Eq. (30) which, translated for the continuum, read

$$m = \mathbf{V}_0\left[\mathbf{r}+\frac{1}{2}\,\mathbf{c}(\mathbf{r})\,\delta t\right]\cdot\hat{\mathbf{n}}_0\left\{1-\left[\frac{\mathbf{c}_0(\mathbf{r})\cdot\hat{\mathbf{n}}_0}{|\mathbf{c}_0(\mathbf{r})|}\right]^2\right\}^{1/2}\ , \tag{39}$$

$$p = |\mathbf{c}_0(\mathbf{r})| - \mathbf{V}_0\left[\mathbf{r}+\frac{1}{2}\,\mathbf{c}(\mathbf{r})\,\delta t\right]\cdot\hat{\mathbf{n}}_0\,\frac{\mathbf{c}_0(\mathbf{r})\cdot\hat{\mathbf{n}}_0}{|\mathbf{c}_0(\mathbf{r})|}\ , \tag{40}$$

$$q = c_0\left[\mathbf{r}+\delta t\,\left(\mathbf{u}(\mathbf{r})+\mathbf{c}_0(\mathbf{r})\right)\right]\left\{1-\left[\frac{\mathbf{c}_0(\mathbf{r})\cdot\hat{\mathbf{n}}_0}{|\mathbf{c}_0(\mathbf{r})|}\right]^2\right\}^{1/2}\ , \tag{41}$$

where

$$\begin{aligned}\mathbf{V}_0\left(\mathbf{r}+\frac{1}{2}\,\mathbf{c}\,\delta t\right) &= \left[1-\mathbf{u}(\mathbf{r})\cdot\mathbf{V}\left(\mathbf{r}+\frac{1}{2}\,\mathbf{c}\,\delta t\right)/c^2\right]^{-1}\left\{\mathbf{V}\left(\mathbf{r}+\frac{1}{2}\,\mathbf{c}\,\delta t\right)\cdot\hat{\mathbf{u}}(\mathbf{r})(1-\gamma_1^{-1})\,\hat{\mathbf{u}}(\mathbf{r})\right.\\ &\quad \left.+\gamma_1^{-1}\ \mathbf{V}\left(\mathbf{r}+\frac{1}{2}\,\mathbf{c}\,\delta t\right)-\mathbf{u}(\mathbf{r})]\right\}\ .\end{aligned} \tag{42}$$

being $\mathbf{V}\left[\mathbf{r}+\mathbf{c}(\mathbf{r})\,\delta t/2\right]$ the velocity of the interface (which must be assigned).

The effective displacement $\delta\mathbf{r}$ is still expressed by Eq. (16), while we must use a relativistic composition for the velocities

$$\begin{aligned}\mathbf{c}[\mathbf{r}+\mathbf{c}(\mathbf{r})\,\delta t] &= \left[1+\mathbf{u}(\mathbf{r})\cdot\mathbf{c}_0^*(\mathbf{r}+\mathbf{c}\,\delta t)/c^2\right]^{-1}\left[\mathbf{c}_0^*(\mathbf{r}+\mathbf{c}\,\delta t)\cdot\,\hat{\mathbf{u}}(\mathbf{r})\,(1-\gamma_1^{-1})\,\hat{\mathbf{u}}(\mathbf{r})\right.\\ &\quad\left.+\,\gamma_1^{-1}\,\mathbf{c}_0^*(\mathbf{r}+\mathbf{c}\,\delta t)+\mathbf{u}(\mathbf{r})\right]\ ,\end{aligned} \tag{43}$$

where $\mathbf{u}(\mathbf{r})$ stand for $\mathbf{u}_1$, $\gamma_1^{-1} = [1-u^2(\mathbf{r})/c^2]^{1/2}$, and $\mathbf{c}_0^*(\mathbf{r}+\mathbf{c}\,\delta t)$ stands for $\mathbf{c}_{02}^*$, given by Eq. (33). In the latter, we must now use the notation for a generally moving fluid writing $\mathbf{c}_{02} = \mathbf{c}_0(\mathbf{r}+\mathbf{c}\,\delta t)$ and

$$\begin{aligned}\mathbf{u}_0(\mathbf{r}+\mathbf{c}\,\delta t) &= \left[1-\mathbf{u}(\mathbf{r})\cdot\mathbf{u}(\mathbf{r}+\mathbf{c}\,\delta t)/c^2\right]^{-1}\left\{\mathbf{u}(\mathbf{r}+\mathbf{c}\,\delta t)\cdot\hat{\mathbf{u}}(\mathbf{r})(1-\gamma_1^{-1})\,\hat{\mathbf{u}}(\mathbf{r})\right.\\ &\quad\left.+\gamma_1^{-1}\ \mathbf{u}(\mathbf{r}+\mathbf{c}\,\delta t)-\mathbf{u}(\mathbf{r})]\right\}\ .\end{aligned} \tag{44}$$

One can therefore obtain the trajectory of a wave beam by a computer since $\mathbf{u}(\mathbf{r}+\delta\mathbf{r})$ and $|\mathbf{c}_0(\mathbf{r}+\delta\mathbf{r})|$ become the initial known values of the subsequent step. Finally, step by step, the computer gives the whole trajectory that can easily be visualized.

IV. APPLICATION TO THE GRAVITATIONAL LENSE EFFECT

All the results found in Sec. III are ready to be applied to the gravitational lense effect. Obviously, we must know the $\mathbf{c}_0(\mathbf{r})$ that appears in Eqs. (33)-(42). Its initial direction $\hat{\mathbf{c}}_0(\mathbf{r})$, when the light enter the cloud of gas surrounding either the galaxy or the star that lies between the Earth and the quasar, is practically equal to the line joining the Earth with the considered quasar that produces the lense effect. Then $|\mathbf{c}_0(\mathbf{r})| = c\, n_r(\mathbf{r})$, where n_r is the refraction index of the hydrogen surrounding the receding galaxy or star. To find $n_r(\mathbf{r})$ we resort to the density profiles proposed by some Authors.

A. Density profiles of the hydrogen gas surrounding a star, a galaxy or a galaxy cluster and corresponding refraction indexes n_r

The matter density profiles for a galaxy or a galaxy cluster is often adequately described by the modified King approximation to the isothermal sphere [4]

$$\rho(r) = \rho_0 \left(1 + \frac{r}{r_c}^2\right)^{-3\beta/2} \quad , \tag{45}$$

where ρ_0 is the central density, r_c the core radius of the galaxy or galaxy cluster. The parameter β is $\beta = \mu m_p \sigma_r^2/(KT)$ where m_p is the proton mass, T the gas temperature and σ_r^2 is the line–of-sight velocity dispersion. Typical values for the best observed galaxy clusters [5] are $\beta = 2/3$, $r_c = 250\,\mathrm{kpc}$ and $\rho_0 = 3 \times 10^{14}\,\mathrm{M}_\odot/\mathrm{Mpc}^3$ [6].

The atmosphere of a star decreases almost exponentially from the star photosphere. We can write [7]

$$\rho(r) = \rho_0 \exp(-r/r_H) \tag{46}$$

where r_H is the density scale height, $r_H = KT/\mu m_p g$ and K is the Boltzmann constant, T the atmosphere temperature, μ the mean molecular mass (in atomic units) and g the surface gravity.

The gas density in a star, a galaxy, or a galaxy cluster can vary within a wide range: typical values for the density in the galactic disk is $0.1 \div 100$ particles/cm^3 while in some denser regions, i.e. star forming regions, molecular clouds, etc. we can have $10^4 \div 10^8$ particles/cm^3. In a solar–type star atmosphere the number density is of the order of 10^{17} particles/cm^3 close to the visible layers [8] (the so–called photosphere) while in the extragalactic medium lower than 10^{-12} particles/cm^3 [9].

The refraction index for the spectral line D of sodium in neutral hydrogen is [10]

$$(n_r - 1)_{20} = 1.32 \times 10^{-4} \tag{47}$$

at atmospheric pressure and 293 K, corresponding to a hydrogen numerical density $N_{20} \simeq 1.74 \times 10^{19}\ \mathrm{cm}^{-3}$. Since $n_r - 1$ scales linearly with numerical density N, we get the general relation

$$(n_r - 1) = \frac{\rho(r)}{N_{20}}(n_r - 1)_{20} \ . \tag{48}$$

B. Calculation of the deviation due to refraction

We apply now the refraction law's and the procedure considered in Sec. III to the light emitted by a distant quasar and refracted by the moving gas of an intermediate galaxy to estimate the deviation of light only due to refraction. For simplicity, we approximate the deviating galaxy, seen hedge-on from the Earth, to a uniform disk of hydrogen. Indeed, we have seen in Sec. IV A that the gas density profile decays rather sharply outside the minimum envelope practically containing all the stars of the considered galaxy. Moreover, since the maximum rotational speed of the galactic cloud is much smaller than the receding velocity of the galaxy, the total deviation almost depends only on the maximum variation of the refraction index n_r between inside the galaxy and outside it (beyond a distance $3R$ from the galaxy centre, where R is the galaxy radius in the system S_1 at rest with the galaxy).

We suppose that the galaxy (at a position intermediate between the quasar and the Earth) move with velocity $\mathbf{v} = \boldsymbol{\beta}\, c$ with respect to the Earth system S_0 with the direction parallel to the line joining the quasar with the Earth. Then, the profile of the galaxy because of relativistic contraction of the longitudinal lengths appears as an elliptic disk with the minor axis along $\mathbf{v}$ (see Fig. 2). Taking the x axis parallel to $\mathbf{v}$, the equation of the ellipse in S_0 is

$$\left(\frac{x}{\gamma^{-1}R}\right)^2+\left(\frac{y}{R}\right)^2=1\ , \tag{49}$$

where R is the major semiaxis.

Let $\mathbf{c}_1$ be the initial velocity of the ray of light emitted by the quasar. Its speed is practically equal to that of light in vacuum. We suppose that the direction when the light enter the cloud of gas surrounding either the galaxy or the star is practically equal to the line joining the Earth with the considered quasar. Then, denoted with ξ the angle between $\mathbf{c}_1$ and $-\hat{\mathbf{e}}_x$ (see Fig. 2), we write

$$\mathbf{c}_1=c\,(-\cos\xi\,\hat{\mathbf{e}}_x+\sin\xi\,\hat{\mathbf{e}}_y)\simeq c\,(-\hat{\mathbf{e}}_x+\xi\,\hat{\mathbf{e}}_y)\ , \tag{50}$$

where, to first order approximation, $\sin\xi\simeq\xi$ and $\cos\xi\simeq 1$.

Applying the treatment of Sec. III, we have to consider two subsequent steps: i) the refraction when the light ray enter the galaxy disk (we suppose that there is a discontinuity in the gas density) and ii) the refraction when the ray, passed through the cloud of gas, comes out of galaxy.

In the first step, the light passes from vacuum to the gas cloud (moving with $\mathbf{v}$ with respect to the Earth system). To apply the refraction procedure we have to consider the system at rest with the first fluid. In this case, because the first medium is the vacuum, the system S_0 coincides with Earth's. Then

$$\mathbf{u}_1=0\ ,\quad \mathbf{u}_2=u_2\,\hat{\mathbf{e}}_x=\beta\,c\,\hat{\mathbf{e}}_x\ ,\quad \mathbf{c}_{01}=\mathbf{c}_1\ . \tag{51}$$

With reference to Fig. 2, if the ray light emitted by Q impinges on the galaxy surface in A, having $y=h$, we derive from Eq. (49) $x=\gamma^{-1}\sqrt{R^2-h^2}$. In the system S_1 at rest with fluid 1 ($u_1=0$) we have

$$\hat{\mathbf{n}}_0=\hat{\mathbf{n}}\ ,\quad \mathbf{V}_{01}=\mathbf{V}=\beta c\,\hat{\mathbf{e}}_x\ ,\vartheta_{01}=\vartheta_1\ , \tag{52}$$

then

$$\cos\vartheta_1=\hat{\mathbf{c}}_1\cdot\hat{\mathbf{n}}=-\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_x+\xi\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_y>0\ . \tag{53}$$

The speed of light in medium 2 has a value a bit smaller than in medium 1 (vacuum), i.e., $c_{02}=c(1-\varepsilon)$, with $\varepsilon<1$. Then, the velocity $\mathbf{c}_{02}$ with respect to medium 1 is

$$\mathbf{c}_{02}=c(1-\varepsilon)\,[\hat{\mathbf{n}}\cos\vartheta_{02}+\hat{\boldsymbol{\sigma}}\sin\vartheta_{02}]\ , \tag{54}$$

where ϑ_{02} is obtained by refraction formulae (7) [with s given by Eq. (29) and m, p, and q by Eq. (25)], and $\hat{\boldsymbol{\sigma}}$ is given by Eq. (31). In this case, to first order approximation, we have

$$\begin{aligned}\cos\vartheta_{02}&\simeq\cos\vartheta_{01}+\varepsilon\,\frac{c\sin^2\vartheta_{01}}{c\cos\vartheta_{01}-\mathbf{V}_0\cdot\hat{\mathbf{n}}}=\cos\vartheta_1+\varepsilon\frac{\sin^2\vartheta_1}{\cos\vartheta_1-\beta\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_x}\ ,\\ \sin\vartheta_{02}&\simeq\sin\vartheta_{01}\left(1-\varepsilon\,\frac{c\cos\vartheta_{01}}{c\cos\vartheta_{01}-\mathbf{V}_0\cdot\hat{\mathbf{n}}}\right)=\sin\vartheta_1\left(1-\varepsilon\frac{\cos\vartheta_1}{\cos\vartheta_1-\beta\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_x}\right)\ .\end{aligned} \tag{55}$$

If we use Eq. (53) with first order approximation ($\xi\ll 1$ and $\varepsilon\ll 1$) we derive from Eq. (55)

$$\cos\vartheta_{02}\simeq\cos\vartheta_1+\varepsilon\frac{\sin^2\vartheta_1}{(1+\beta)\cos\vartheta_1}\ , \tag{56}$$

$$\sin\vartheta_{02}\simeq\sin\vartheta_{01}\left(1-\varepsilon\,\frac{1}{1+\beta}\right)\ . \tag{57}$$

Substituting Eqs.(56) and (57) into Eq. (54) we obtain

$$\mathbf{c}_{02}\simeq c\left\{\hat{\mathbf{c}}_1+\frac{\varepsilon}{1+\beta}\left[\frac{\hat{\mathbf{n}}}{\cos\vartheta_1}-(\beta+2)\hat{\mathbf{c}}_1\right]\right\}\ , \tag{58}$$

The effective velocity $\mathbf{c}^*_{02}$ is obtained by relativistically adding $\mathbf{c}_{02}$ to the relative velocity $\mathbf{u}_{02}$ of medium 2 with respect to medium 1. Since $u_1=0$, the velocity $\mathbf{c}_2$ with respect to the system S_0 is equal to $\mathbf{c}^*_{02}$, i.e.,

$$\mathbf{c}_2=\mathbf{c}^*_{02}=c\left[-\hat{\mathbf{e}}_x+\frac{\xi\,\hat{\mathbf{e}}_y}{\gamma(1-\beta)}+\frac{\varepsilon\gamma\hat{\mathbf{n}}}{\cos\vartheta_1}+\varepsilon\gamma\,\hat{\mathbf{e}}_x+\varepsilon\,\hat{\mathbf{e}}_x\frac{1+\beta}{1-\beta}\right]\ . \tag{59}$$

To calculate $\hat{\mathbf{n}}$ we substitute the galaxy profile $f(x,y)$ given by Eq. (49) into Eq. (19) thus obtaining

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}(x,y) = -\frac{\nabla f(x,y)}{|\nabla f(x,y)|} = -\frac{\gamma^2 x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y}{\gamma(R^2-\beta^2 y^2)^{1/2}} \tag{60}$$

We have at point A of Fig. 2

$$\hat{\mathbf{n}}_A = -\frac{\gamma^2\sqrt{R^2-h^2}\,\hat{\mathbf{e}}_x + h\,\hat{\mathbf{e}}_y}{[\gamma^2(R^2-h^2)+h^2]^{1/2}} \ , \tag{61}$$

and Eq. (53) becomes

$$\cos\vartheta_1 \simeq \frac{\gamma\sqrt{R^2-h^2}}{\sqrt{\gamma^2(R^2-h^2)+h^2}} \ . \tag{62}$$

To proceed to the second step, we find the point B of Fig. 2 where the light beam impinges again on the galaxy profile with a second refraction, passing from medium 2 to medium 3, which is again the vacuum. Point B is the intersection between the galaxy profile defined by Eq. (49) and the straight line directed as $\mathbf{c}_2$, expressed by

$$\begin{cases} x - x_A = k\mathbf{c}_2\cdot\hat{\mathbf{e}}_x/c \ , \\ y - y_A = k\mathbf{c}_2\cdot\hat{\mathbf{e}}_y/c \ . \end{cases} \tag{63}$$

Solving the system of two equations (ellipse and straight line), we obtain

$$\begin{aligned} y_B &= h + 2\sqrt{R^2-h^2}\left[\xi(1+\beta) - \varepsilon\frac{h}{\gamma\sqrt{R^2-h^2}}\right] \ , \\ x_B &= \gamma^{-1}\left\{-\sqrt{R^2-h^2} + 2h\left[\xi(1+\beta) - \varepsilon\frac{h}{\gamma\sqrt{R^2-h^2}}\right]\right\} \ , \end{aligned} \tag{64}$$

In the system $S_2 = S_0$ at rest with the galaxy, the latter is observed as a circular disk so that the unit normal $\hat{\mathbf{n}}_0'$ in B is (see Fig. 2)

$$\hat{\mathbf{n}}_0' = \frac{x_B\,\hat{\mathbf{e}}_x + y_B\,\hat{\mathbf{e}}_y}{R} \ , \tag{65}$$

where x_B and y_B are given by Eq. (64). We derive from Eqs. (2) and (58) [it is $|\mathbf{c}_{02}| = c(1-\varepsilon)$, as clearly appears from Eq. (54)]

$$\cos\vartheta_{02} = \hat{\mathbf{c}}_{02}\cdot\hat{\mathbf{n}}_0' = \hat{\mathbf{c}}_1\cdot\hat{\mathbf{n}}_0' + \frac{\varepsilon}{1+\beta}\left(\frac{\hat{\mathbf{n}}_A\cdot\hat{\mathbf{n}}_0'}{\cos\vartheta_1} - \hat{\mathbf{c}}_1\cdot\hat{\mathbf{n}}_0'\right) \ . \tag{66}$$

The use of Eqs. (50), (64), and (65) leads to

$$\hat{\mathbf{c}}_1\cdot\hat{\mathbf{n}}_0' = -\hat{\mathbf{n}}_0'\cdot\hat{\mathbf{e}}_x + \xi\hat{\mathbf{n}}_0'\cdot\hat{\mathbf{e}}_y = \frac{\sqrt{R^2-h^2}}{R} - \frac{2h}{R}\left[\xi(1+\beta) - \varepsilon\frac{h}{\gamma\sqrt{R^2-h^2}}\right] + \xi\frac{h}{R} \ . \tag{67}$$

The use of Eqs. (61), (64), and (65) leads to

$$\begin{aligned} \hat{\mathbf{n}}_A\cdot\hat{\mathbf{n}}_0' &= -\frac{\sqrt{R^2-h^2}\hat{\mathbf{n}}_0'\cdot\hat{\mathbf{e}}_x}{\sqrt{R^2-\beta^2h^2}} - \frac{h\hat{\mathbf{n}}_0'\cdot\hat{\mathbf{e}}_y}{\gamma\sqrt{R^2-\beta^2h^2}} \\ &= -\frac{1}{R\sqrt{R^2-\beta^2h^2}}\left\{\gamma^{-1}h^2 - (R^2-h^2)\right. \\ &\quad \left. + 2h\sqrt{R^2-h^2}(1+\gamma^{-1})\left[\xi(1+\beta) - \varepsilon\frac{h}{\gamma\sqrt{R^2-h^2}}\right]\right\} \ . \end{aligned} \tag{68}$$

Substituting Eqs. (67) and (68) into Eq. (66), we obtain

$$\begin{aligned} \cos\vartheta_{02} &= \sqrt{1-\frac{h^2}{R^2}} + \xi\frac{h}{R} - \frac{2h}{R}\left[\xi(1+\beta) - \varepsilon\frac{h}{\gamma\sqrt{R^2-h^2}}\right] - \frac{\varepsilon\gamma^{-1}h^2}{(1+\beta)R\sqrt{R^2-h^2}} = \\ &= \sqrt{1-\frac{h^2}{R^2}} - \xi\frac{h}{R}(1+2\beta) + \frac{\varepsilon\gamma^{-1}h^2(1+2\beta)}{(1+\beta)R\sqrt{R^2-h^2}} \end{aligned} \tag{69}$$

We now apply the refraction formulae (7), taking into account that in this case it is $m = 0$, $p = c'_{01} = c_{02} = c(1-\varepsilon)$, and $q = c'_{02}\sin\vartheta'_{01} = c\sin\vartheta_{02}$. We obtain (putting an apex in order to parallel the same notation of Sec. III)

$$\sin\vartheta'_{02} = \frac{c'_{02}}{c'_{01}}\sin\vartheta'_{01} = \sin\vartheta_{02}(1+\varepsilon)\ , \tag{70}$$

$$\cos\vartheta'_{02} = \sqrt{1-\sin^2\vartheta'_{02}} \simeq \cos\vartheta_{02} - \varepsilon\frac{\sin^2\vartheta_{02}}{\cos\vartheta_{02}}\ . \tag{71}$$

The velocity $\mathbf{c}'_{02}$ after the second refraction is

$$\begin{aligned}\mathbf{c}'_{02} &= c\left[\hat{\mathbf{n}}'_0\cos\vartheta'_{02} + \frac{\mathbf{c}_{02} - \hat{\mathbf{n}}'_0\cos\vartheta_{02}}{\sin\vartheta_{02}}\sin\vartheta'_{02}\right] \\ &\simeq c\left[\hat{\mathbf{c}}_1(1+\varepsilon) + \frac{\varepsilon}{1+\beta}\left(\frac{\hat{\mathbf{n}}_A}{\cos\vartheta_1} - \hat{\mathbf{c}}_1\right) - \frac{\varepsilon\hat{\mathbf{n}}'_0}{\cos\vartheta'_{01}}\right]\ .\end{aligned} \tag{72}$$

To find $\mathbf{c}'^{*}_{02}$, we relativistically add $\mathbf{c}'_{02}$ to the relative velocity $\mathbf{u}'_{02} = -\mathbf{u}_2$. The velocity $\mathbf{c}'_2$ in S_0 is then obtained adding $\mathbf{u}_2$ to the velocity $\mathbf{c}'^{*}_{02}$. But, in this way, we find again $\mathbf{c}'_2 = \mathbf{c}'_{02}$. Finally, we calculate the angle δ of Fig. 2, given by

$$\sin\delta = \hat{\mathbf{c}}'_2\cdot\hat{\mathbf{e}}_y \simeq \xi - \frac{\varepsilon h}{\sqrt{R^2-h^2}}\left(\sqrt{\frac{1-\beta}{1+\beta}} + 1\right)\ . \tag{73}$$

To give a figure to the deviation angle $\delta - \xi$ expressed by Eq. (73) we consider a deviating galaxy receding with $\beta = 0.6$. Remembering that we introduced ε as $c_{02} = c/n_r = c(1-\varepsilon)$, we derive from Eqs. (47) and (48)

$$1 - \varepsilon = \left[1 + 1.32\times 10^{-4}\frac{\rho(r)}{N_{20}}\right]^{-1}\ . \tag{74}$$

Having schematized the galaxy with an abrupt variation of density, we take $r \simeq 0$ in Eq. (45), since what matters is the maximum variation of c_{02} (or of the refraction index n_r) from outside to inside the galaxy. Consequently, we take $\rho(r) = \rho_0$ in Eq. (74), and for ρ_0 the maximum value reported in Sec. IV A, i.e., $\rho_0 = 10^8/\text{cm}^3$. Since $N_{20} = 1.74\times 10^{17}$ cm^{-3}, we derive from Eq. (74)

$$\varepsilon \simeq 10^{-13}\ . \tag{75}$$

The maximum deviation is obtained by a grazing beam of light as appears from Eq. (73). Taking into account that the light comes from a quasar with $\beta_q \simeq 0.9$ and that a minimum spreading of density at the edge of the galaxy has to be considered, a realistic value is $h = 0.9999R$. By this value, $\beta = 0.6$, and the use of Eqs. (73) and (75) we finally obtain

$$\text{deviation} = \delta - \xi \simeq \sin\delta - \xi = -10^{-11}\ \text{rad} \simeq -2\times 10^{-6}\ ''\ , \tag{76}$$

the negative sign implying a converging effect similar to the gravitational one.

To be an appreciable fraction of standard gravitational deviations, we require at least $\delta - \xi \simeq -10^{-6}\ ''$, i.e., a value 5×10^{-3} times that given by Eq. (76). The deviation due to refraction because of the intergalactic gas is therefore negligible.

C. Refraction due to a star atmosphere

The only interesting case could be the refraction provided by the close passage to a star in a galaxy seen hedge-on from the Earth. However, even if in this case the gas density is higher by several orders of magnitudes with respect to the interstellar or intergalactic medium, the probability of such an event turns out to be small. This probability can be estimated comparing the effective radius for refraction for all the stars in a galaxy to the total axial cross-section of the hedge-on seen galaxy. If we assume, to be conservative, that stars do not screen each other in a galaxy, the probability p can be estimated with:

$$p \simeq \frac{N_*\pi R^2_{\text{eff}}}{\text{galaxy cross-section}}\ , \tag{77}$$

where N_* is the number of stars in the considered galaxy and $R_{\rm eff}$ is the effective radius of the stars to provide a sensitive contribution to the refraction. The galaxy area can be estimated assuming the most favourable case: a spiral galaxy with no dust seen edge–on. Taking the Milky Way as a reference the region with sensitive star density can be approximated with a box 5 kpc × 1 kpc wide. These figures can be assumed for a very wide range of galactic structures. The number of stars can be $\sim 10^{11}$. The average radius of a star is somewhat lower than the solar radius, $R_\odot \sim 7 \times 10^{10}$ cm but, to be conservative we assume that the maximum distance from the star to feel a sensitive refraction is ~ 50 times the stellar radius. From Eq. (77) the probability turns out to be $\sim 10^{-7}$, i.e. negligible for any realistic computation. Even considering only the densest stellar regions, i.e. the cores of globular clusters, we can have 10^6 stars in a region 10 pc wide and the probability turns out to be of the same order of magnitude.

V. CONCLUSIONS

In Sec. II we extended the results of Ref. [1] to a regularly moving medium. This extension will be applied in a future paper to the motion of filaments [3]. In Sec. III we have found the exact expressions for the refraction of waves traversing two media in relativistic motion. This result has been extended in Sec. III D to inhomogeneous, regularly moving media.

The application to the lense effect, in order to see possible corrections to the main gravitational deviation, has been performed in Sec. IV B. Considering a receding galaxy seen hege-on from the Earth, the calculations leads to Eq. (76) that yealds a negative deviation (it is a converging effect) $|\delta - \xi| \simeq 10^{-11}$ rad $\simeq 2 \times 10^{-6}$ ″, which is less than 10^{-3} times the minimum observable correction 10^{-2} ″ to the gravitational lense effect. We can therefore garantee to astronomers that the additional deviation due to refraction by part of the intergalactic gas is negligible.

In Sec. IV C we have evaluated the probability p that one of the two light beams constituting the lense effect is appreciably deviated by the atmosphere of a star belonging to the considered receding galaxy. The result turn out to be $p \simeq 10^{-7}$ so that a consistent deviation due to refraction is quite small.

Concluding, the additional deviation due to relativistic refraction of the intergalactic gas is negligible, and possible deviations of some arc seconds due to stellar atmospheres is a very rare event.

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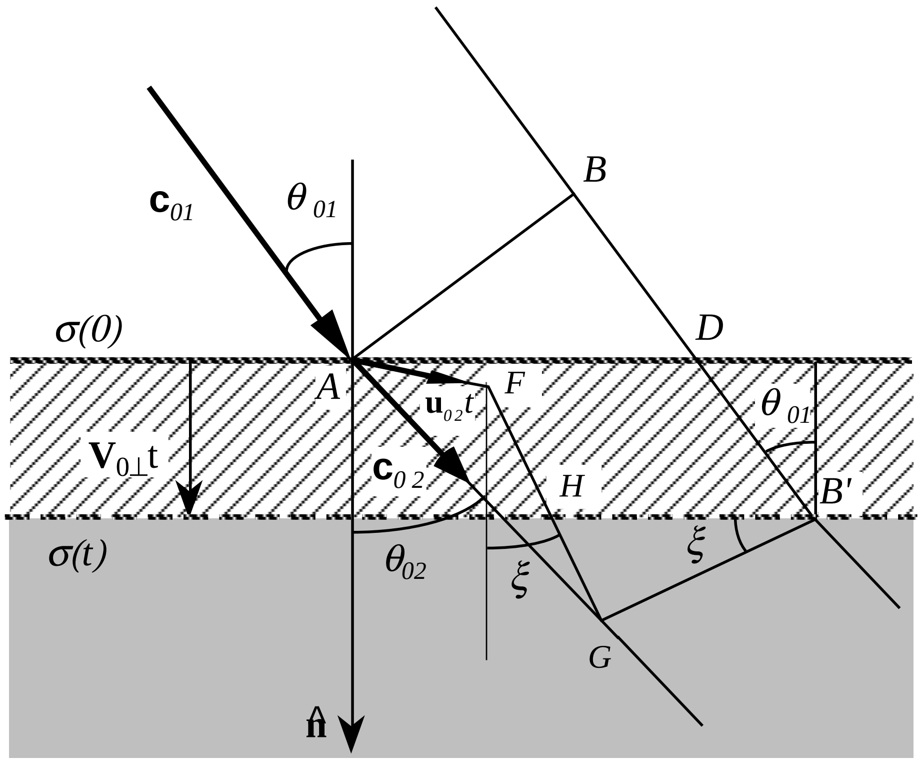

FIG. 1. A nonrelativistic wave has velocity $\mathbf{c}_{01}$ in medium 1 and equiphase surface AB perpendicular to $\mathbf{c}_{01}$ if the observer S_0 is at rest with medium 1. An interface σ, having local velocity $\mathbf{V}_0$, separates medium 1 from medium 2. At time 0 medium 2 includes both the streaked region and the shaded region. At time t, when B reaches B', medium 2 is represented by the shaded area only. The streaked region is the one swept by the interface σ from time 0 to time t. (Obviously, if $V_0 = V_{0\perp} = 0$ the streaked region is absent $D = B'$). Consequently, during the time interval between 0 and t ray AA' travels only in medium 2 while ray BB' travels only in medium 1. $\hat{\mathbf{n}}$ is the unit vector perpendicular to the interface, chosen so that $\mathbf{c}_{01} \cdot \hat{\mathbf{n}} > 0$. When a wave ray impinges on the interface in A the wave is refracted in medium 2 with velocity $\mathbf{c}_{02}$. Point B of the wave front reaches the moving interface in B'. The wave front GB' of the refracted wave is obtained as the envelope of the spherical waves radiated by the points of the interface consecutively reached by the impinging wave front. The spherical wave radiated by A has its center moving with velocity $\mathbf{u}_{02}$ and reaching F at time t. Its radius at time t is FG. Since $\mathbf{u}_{02}$ does not lie, in general, in the incidence plane ζ (that is the plane of the figure), points B, D, B', and G lie in a plane parallel to ζ and containing F. In the general case of media moving with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, we add $\mathbf{u}_1$ to $\mathbf{c}_{01}$ and $\mathbf{u}_2$ to $\mathbf{c}_{02}$.

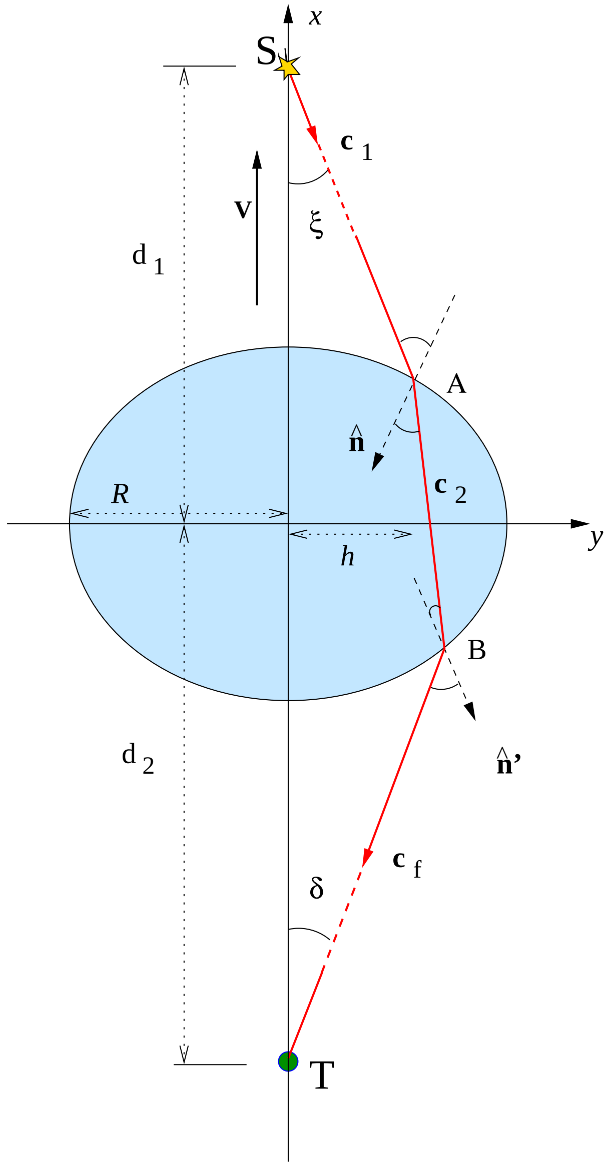

FIG. 2. A light beam coming from a quasar S, for instance receding with $\beta = 0.9$, impinges at A on a galaxy, for instance receding with $\beta = 0.6$. Considering separately the deviation due to gravitation, the light beam is refracted in A and comes out at B. If ξ and δ are the angles of the light beam with the straight line x joining the Earth T with S, the deviation of the light beam is given by $\delta - \xi$.