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# Review of Problems of Dynamics in the Hyperbolic Theory of Special Relativity

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## 1. **Abstract**

The present paper considers some ideas relevant to the formulation of the equations of dynamics in the hyperbolic theory of special relativity. After a brief introduction on the Sommerfeld spherical representation and the associated hyperbolic representation the paper treats two main topics, the Newtonian Second Law of Motion and the calculation of the Thomas-Wigner rotation angle.

The original interpretation of Newton's law of motion in Special Relativity by Lorentz and Einstein made use of the concept of velocity dependent mass and this interpretation has since become part of the accepted standard theory. There is an alternative view however expressed in the hyperbolic theory by the equation

*force = rest mass x hyperbolic acceleration* (1.1)

where there is no velocity dependent mass. The velocity dependence occurs but it is in the acceleration and not in the mass term. What is normally overlooked is that relativistic acceleration is not expressible by the familiar classical formula since acceleration relates to velocity increase which must be calculated according to Einstein's composition rule and not by the simple subtraction of classical physics. So the hyperbolic acceleration formulation involves nothing else than acceleration correctly interpreted according to the rules of Special Relativity.

The Thomas-Wigner rotation angle associated with the product of Lorentz transformations makes its appearance in the Thomas rotation associated with three dimensional particle motion and in the collision calculations of atomic physics. The difficulty of the calculation of this angle is well known. However, in the Sommerfeld or hyperbolic view the angle finds a simple geometric interpretation as spherical excess or hyperbolic defect of the triangle of velocities and can be found immediately using known trigonometric formulae.

## 2. **The Sommerfeld Spherical Representation**.

The hyperbolic theory of special relativity is directly related to the more intuitive but less precise spherical representation due to Sommerfeld (1909, 1932) which is well described in Belloni & Reina (1986). Minkowski had represented the Lorentz transformation in the form of a Euclidean rotation

$$x' = x \cos\varphi + ict \sin\varphi$$
$$ict' = -x \sin\varphi + ict \cos\varphi \qquad (2.1)$$

by using the purely imaginary angle $\varphi$ defined by

$$\tan\varphi = iv/c \qquad (2.2)$$

which results in the identities

$$\cos\varphi = 1/\sqrt{(1- v^2/c^2)}, \ \sin\varphi = i(v/c)/\sqrt{(1- v^2/c^2)} \qquad (2.3)$$

This representation shows clearly the additivity of $\varphi$ with respect to successive transformations.

Sommerfeld (1909) extended this idea to derive Einstein's composition of non-rectilinear velocities by representing Lorentz transformations by arcs on the surface of a sphere of imaginary radius. Suppose that $\varphi_1$, $\varphi_2$ are the Minkowski angles corresponding to velocities $\mathbf{v_1}, \mathbf{v_2}$. Then the Minkowski angle $\varphi$ for the resultant velocity $\mathbf{v}$ is given by

$$\cos\varphi = \cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2 \cos\theta \qquad (2.4)$$

Here $\theta$ is the angle between the two velocities. This equation is just the cosine rule for spherical triangles expressed in terms of $\theta$ which is the complement of the corner angle of the triangle of velocities. The general composition law then follows by using relations of the type (2.3) from which we get

$$\frac{1}{\sqrt{(1 - v^2/c^2)}} = \frac{(1 + v_1/c.v_2/c \cos\theta)}{\sqrt{(1 - v_1^2/c^2)(1 - v_2^2/c^2)}} \qquad (2.5)$$

Then inversion and squaring gives

$$(1 - v^2/c^2) = \frac{(1 - v_1^2/c^2)(1 - v_2^2/c^2)}{(1 + v_1/c.v_2/c \cos\theta)^2} \qquad (2.6)$$

Solving for v gives the composition law

$$v = \frac{\sqrt{\{v_1^2 + v_2^2 + 2 v_1 v_2 \cos\theta - (v_1 v_2/c. \sin\theta)^2\}}}{1 + (v_1.v_2/c^2) \cos\theta} \qquad (2.7)$$

for the magnitude of both the non-commutative sums $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_2 + \mathbf{v}_1$ (i.e. $\mathbf{v}_1$ followed by $\mathbf{v}_2$ and $\mathbf{v}_2$ followed by $\mathbf{v}_1$ respectively) For collinear velocities $\theta$ is zero and there follows the scalar law

$$v = \frac{v_1 + v_2}{1 + v_1.v_2/c^2} \qquad (2.8)$$

## 3. **The Hyperbolic Theory**

The hyperbolic theory comes about by representing the Minkowski's imaginary angle $\varphi$ as iw where w, the rapidity, is real and defined by

$$\text{th } w = v/c \qquad (3.1)$$

giving

$$\text{ch } w = 1/\sqrt{(1-v^2/c^2)}, \quad \text{sh } w = (v/c)/\sqrt{(1-v^2/c^2)} \qquad (3.2)$$

The Lorentz transformation then takes the symmetric form

$$\begin{aligned} ct' &= ct \text{ ch } w - x \text{ sh } w \\ x' &= - ct \text{ sh } w + x \text{ ch } w \end{aligned} \qquad (3.3)$$

This was used by Varićak (1910, 1912) and later by Whittaker (1952). The linear composition of rapidities $w_1$ and $w_2$ to give w follows from

$$\text{th } (w_1+w_2) = \frac{\text{th } w_1 + \text{th } w_2}{1 + \text{th } w_1 . \text{th } w_2} \qquad (3.4)$$

The vector addition of rapidities follows as in the Sommerfeld method by putting

$$\varphi_1 = iw_1, \quad \varphi_2 = iw_2 \qquad (3.5)$$

to give

$$\text{ch } w = \text{ch } w_1 \text{ ch } w_2 + \text{sh } w_1 \text{ sh } w_2 \cos \theta \qquad (3.6)$$

- a transformation due to Varićak (1910). Expressed in terms of the interior angle $\pi-\theta$ it is the cosine rule in hyperbolic space.

Instead of rapidity we may equally use the hyperbolic velocity

$$V = c\, w = c \text{ th}^{[-1]} (v/c) = v \{1 + (v/c)^2/3 + ... \} \qquad (3.7)$$

This retains the additivity of rapidity and approximates ordinary velocity v when $v \ll c$.
The addition rule for hyperbolic velocities $V_1$, $V_2$ inclined at an angle θ to give V becomes

$$\text{ch } V/c = \text{ch } V_1/c \text{ ch } V_2/c + \text{sh } V_1/c \text{ sh } V_2/c \cos \theta \qquad (3.8)$$

The transition to hyperbolic space makes only a slight change to the mathematics but makes a great difference in the way in which special relativity is viewed since it recognises a physical reality to hyperbolic geometry which has the reputation of being an abstract purely mathematical construction.

Works in the hyperbolic theory in the literature are relatively few. The main contribution is due to Varićak (1910, 1912 etc.) summarized in his book (1924). Of significance also was the work of Borel (e.g. 1913) who saw clearly the consequences of a non-Euclidean kinematic space. Later Fock (1955) interpreted relative velocity using hyperbolic geometry. The hyperbolic interpretation in a somewhat different form was later used in particle physics by Smorodinski (1963) and was mentioned by Wick (1963) and others. The writer has been contributing to the theory at these PIRT conferences since 1994.

4. **Hyperbolic Acceleration**:

This is defined as rate of change of hyperbolic velocity V with respect to proper time $\tau$. It will be denoted by $\alpha$:

$$\alpha = \frac{dV}{d\tau} \tag{4.1}$$

The proper time $\tau$ here is time observed relative to the moving body.

In spite of its appearance, hyperbolic acceleration is just normal acceleration relativistically defined, the familiar expressions dv/dt, $d^2x/dt^2$ being only valid in the frame of the moving body. In any time increment $\delta t$, the increase in velocity $\delta v$ of an accelerating body will not occur at a fixed position x but will take place over a certain distance $\delta x$. The increase of velocity relative to the origin should consequently be calculated by Einstein's formula as

$$\frac{(v+\delta v)-v}{1 - (v+\delta v).v/c^2} = \frac{\delta v}{1 - (v+\delta v).v/c^2}. \tag{4.2}$$

which gives, as $\delta t \rightarrow 0$ and $\delta v \rightarrow 0$, a first order increment of

$$\frac{\delta v}{1 - v^2/c^2} = c\,\delta\{ th^{[-1]}\, v/c \} = \delta V \tag{4.3}$$

Taking account of time change from the origin to the moving particle, the acceleration is found as

$$\alpha = \frac{dv}{d\tau}.\frac{1}{1 - v^2/c^2} = \frac{dV}{d\tau} \tag{4.4}$$

which may also be written

$$\alpha = \frac{1}{(1-(v/c)^2)^{3/2}}\frac{dv}{dt} \tag{4.5}$$

*EXAMPLE*

The relativistic motion of a particle undergoing constant acceleration $\alpha$ in its own moving frame is was called 'hyperbolic' by Born (1909) because with a suitable origin of coordinates the equation relating distance x and time t can be written

$$x^2 - (ct)^2 = A^2 \tag{4.6}$$

where A is the constant $c^2/\alpha$. The parametrization

$$x = A\,ch\,u, \quad ct = A\,sh\,u \tag{4.7}$$

then leads to

$$dx/dt = c \, \text{th} \, u \qquad (4.8)$$

$$c \, d\tau = \sqrt{(c^2 dt^2 - dx^2)} = A \, du \qquad (4.9)$$

Equation (4.8) identifies u with rapidity so the hyperbolic acceleration is

$$dV/d\tau = c \, du/d\tau = c^2/A = \alpha \qquad (4.10)$$

verifying that hyperbolic acceleration coincides with acceleration referred to the moving particle.

The representation of the motion here contrasts with that arising from a similar analysis in the Minkowski complex representation as presented e.g. by Sommerfeld in his 1923 notes which he added to Minkowski's 1908 article 'Space and Time' (See pp. 94-95 in *The Principle of Relativity* reprint). The equation (4.6) is there written in the form

$$x^2 + (ict)^2 = A^2 \qquad (4.11)$$

and from this we get the picture of a constant speed circular motion in the space of the variables x, ict with a centrifugal acceleration equal to the value of acceleration here found for α.

## 5. **Rectilinear Mechanics**

Planck (1906), commenting on Einstein (1905), showed that Newton's law of classical dynamics in the form

$$F = \frac{dp}{dt} \qquad (5.1)$$

remains valid in Special Relativity if momentum is defined as

$$p = \frac{mv}{\sqrt{(1 - v^2/c^2)}} \qquad (5.2)$$

m here denoting what is usually called rest mass which becomes the velocity dependent mass after division by the root term. Leaving aside this interpretation, Planck’s equation (5.2) may easily be transformed into a form involving hyperbolic acceleration, for we have

$$dp/dt = m \, d/dt \, \{v/\sqrt{(1 - v^2/c^2)}\}$$

$$= m \, (1 - v^2/c^2)^{-3/2} \, dv/dt$$

$$= mc \, (1 - v^2/c^2)^{-1/2} \, d/dt \, \text{th}^{[-1]}(v/c)$$

$$= m \, dV/d\tau \qquad (5.3)$$

Newton's law for rectilinear motion then takes the form

$$F = m \frac{dV}{d\tau} = m \, \alpha \qquad (5.4)$$

i.e.

*force = rest mass x hyperbolic acceleration* (5.5)

Alternatively, following the writer's PIRT 2000 conference paper, starting from

$$p = mc \, sh \, V/c \tag{5.6}$$

we get

$$dp/dt = m \, ch \, V/c. \, dV/dt$$
$$= m/\sqrt{(1 - v^2/c^2)} \; dV/dt$$
$$= m \, dV/d\tau \tag{5.7}$$

This again gives the Newton law.

If F is constant, hyperbolic acceleration is constant with the value F/m resulting in Born's hyperbolic motion.

## 6. **Three Dimensional Dynamics**

The equations of three dimensional motion of a particle as formulated by Planck (1906) and Einstein (1907) can be written in vector form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{6.1}$$

where **p** is a vector having components

$$p_x = \frac{mv_x}{\sqrt{(1 - v^2/c^2)}}$$

$$p_y = \frac{mv_y}{\sqrt{(1 - v^2/c^2)}}$$

$$p_z = \frac{mv_z}{\sqrt{(1 - v^2/c^2)}} \tag{6.2}$$

which may be put in the form

$$\mathbf{p} = p \, \mathbf{n} \tag{6.3}$$

where p denotes the scalar value and **n** is a unit vector in the direction of v. The equations of motion become

$$\mathbf{F} = \frac{dp}{dt}.\mathbf{n} + p.\frac{d\mathbf{n}}{dt} \tag{6.4}$$

On multiplying scalarly by **n** the last term vanishes since d**n**/dt is perpendicular to **n**. So

$$\mathbf{F.n} = \frac{dp}{dt} \quad (6.5)$$

On the left hand side is the tangential component of force while on the right we have the same expression as in the rectilinear case. Consequently we deduce:

*tangential force = rest mass x tangential hyperbolic acceleration* (6.6)

Written out this equation is

$$\mathbf{F.n} = \frac{m}{\{1 - v^2/c^2\}^{3/2}} \frac{dv}{dt} \quad (6.7)$$

Here again the coefficient of dv/dt on the right hand side has in the past been known as the velocity dependent longitudinal mass but in the hyperbolic theory the velocity dependence is seen as an essential part of the expression for acceleration. The general analysis of the three dimensional motion is complicated by the occurrence of the Thomas rotation.

## 7. **Product of Lorentz Transformations**

The discussion of three - dimensional kinematics in special relativity requires an analysis of the product of two pure Lorentz transformations. It is well known that such a product involves a spatial rotation so that we may write

$$\Lambda_2.\Lambda_1 = R\,\Lambda \quad (7.1)$$

where R is a spatial rotation matrix and $\Lambda$ a pure Lorentz transformation resulting from the composition of $\Lambda_1$ followed by $\Lambda_2$. Concerning the explicit calculation of the matrix R we find it stated, even as late as 1980, in the well known text of Goldstein:

> *"The algebra is quite forbidding, more than enough, usually, to discourage any actual demonstration of the rotation matrix".*

Actually, the problem of finding the rotation matrix had already been analysed in the 1914 book by Silberstein. From his analysis it follows that R is a rotation about an axis perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$ through an angle equal to the angle between the two non - commutative compositions $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_2+\mathbf{v}_1$. This important observation can be used to determine the rotation angle.

*EXAMPLE 1*: Rotation angle for velocities at right angles.

The Einstein composition formula gives for the squared magnitude of both $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_2+\mathbf{v}_1$

$$v^2 = v_1^2 + v_2^2 - (v_1.v_2/c)^2 \quad (7.2)$$

This may be put in the two forms:

$$v^2 = v_1^2 + v_2^2(1 - v_1^2/c^2) = v_1^2(1 - v_2^2/c^2) + v_2^2 \quad (7.3)$$

These correspond to the two velocity compositions which have Cartesian forms

$$(v_1,\ v_2\sqrt{(1 - v_1^2/c^2)}),\ \ (v_1\sqrt{(1 - v_2^2/c^2)},\ v_2), \tag{7.4}$$

By taking the vector product we find, for the sine of the angle ψ between these,

$$\sin\psi = \frac{v_1.v_2}{v^2}.\{1 - \sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}\} \tag{7.5}$$

$$= \frac{v_1.v_2}{c^2}.\frac{1}{\{1 + \sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}\}} \tag{7.6}$$

and, using the factors

$$\gamma_1 = 1/\sqrt{(1 - v_1^2/c^2)},\ \ \gamma_2 = 1/\sqrt{(1 - v_2^2/c^2)} \tag{7.7}$$

the resulting expression for the rotation angle may be written

$$\sin\psi = \frac{v_1 v_2}{v^2}.\frac{\{\gamma_1.\gamma_2 - 1\}}{\gamma_1\,\gamma_2} = \frac{v_1 v_2}{c^2}.\frac{\gamma_1\,\gamma_2}{\{1 + \gamma_1\,\gamma_2\}} \tag{7.8}$$

*EXAMPLE 2*: Rotation angle for inclined velocities.

Finding the rotation angle ψ for the combination of two velocities $v_1$, $v_2$ inclined at a general angle θ is much more difficult. The value of ψ was first given in the paper of Ben-Menahem (1985) whose analysis appears to be based directly on Silberstein's although algebraic steps were left unexplained leaving details unclear. The formula given for the half angle ψ/2 was

$$\tan\psi/2 = \frac{\sin\theta}{(k + \cos\theta)} \tag{7.9}$$

k being positive and satisfying

$$k^2 = \frac{(\gamma_1 + 1)(\gamma_2 + 1)}{(\gamma_1 - 1)(\gamma_2 - 1)} \tag{7.10}$$

For velocities at right angles, cos θ = 0, sin θ = 1 and (7.9) becomes

$$\tan\psi/2 = 1/k \tag{7.11}$$

from which

$$\sin\psi = \frac{2\tan\psi/2}{1 + \tan^2\psi/2} = \frac{2k}{k^2 + 1}. \tag{7.12}$$

which is easily shown to agree with the value found above.

For a general angle θ we find

$$\sin\psi = \frac{2\tan\theta/2}{1 + \tan^2\theta/2} = \frac{2\sin\theta\,(k + \cos\theta)}{(k + \cos\theta)^2 + \sin^2\theta} \tag{7.13}$$

$$\cos\psi = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{(k + \cos\theta)^2 - \sin^2\theta}{(k + \cos\theta)^2 + \sin^2\theta} \qquad (7.14)$$

$$\tan\psi = \frac{2\tan\theta/2}{1 - \tan^2\theta/2} = \frac{2.\sin\theta\,(k + \cos\theta)}{(k + \cos\theta)^2 + \sin^2\theta} \qquad (7.15)$$

Such equations were quoted by Ungar (1988) who however gave no details on the derivation saying only that it involves considerable difficulty. On the other hand, these equations may easily be derived by non-Euclidean geometry as described below using only well established trigonometric formulae.

## 8. **The Thomas Precession**

The rotation angle became of interest to physicists following Thomas' 1926 paper on the precessing electron. The Thomas precession occurs when acceleration **f** is in a different direction to velocity **v** so that at two successive instants t, t + δt the velocities **v**, **v** + **f** δt do not have the same direction. Suppose the velocity increment **f** δt is decomposed into components parallel and orthogonal to **v**. Since the increments are infinitesimal, they may be superimposed so the resulting spatial rotation arises only from the orthogonal component and may be calculated using the result of the last section by putting

$$v_1 = v, \ v_2 = f.\delta t \ \sin\theta \qquad (8.1)$$

where θ is the inclination to **f** to **v**. Using (7.8) with

$$\gamma_1 = 1/\sqrt{(1 - v^2/c^2)} = \gamma, \ \gamma_2 = 1 \qquad (8.2)$$

the magnitude of the infinitesimal rotation angle δE is found as

$$\delta E = \sin\delta E = \frac{(\gamma-1)}{\gamma} \ v\,f\,\delta t \ \sin\theta \qquad (8.3)$$

giving a rotation rate

$$\Omega = \frac{(\gamma-1)}{\gamma v^2}. \ \mathbf{v} \times \mathbf{f} \qquad (8.4)$$

correct in both magnitude and direction. When v<<c this may be approximated

$$\Omega = \mathbf{v} \times \mathbf{f}/2c^2 \qquad (8.5)$$

which is the expression given in Thomas's first paper of 1926. In his second paper of 1927 he derived the slightly more general formula (8.4)

## 9. **Determination of Rotation Angle by Non-Euclidean Geometry.**

Using his spherical representation of Lorentz transformations Sommerfeld (1932) found a new method for deriving the Thomas rotation formula. He showed that the rotation angle is equal to the spherical excess of the spherical triangle of velocities i.e. the angle E by which the sum of the angles A, B, C of this triangle exceeds $\pi$:

$$E = (A+B+C) - \pi \tag{9.1}$$

The calculation of rotation angle is thus reduced to the geometrical problem of finding the spherical excess of this triangle. See Sommerfeld (1952), Belloni & Reina (1986).

The use of imaginary quantities in Sommerfeld's method is avoided in the corresponding hyperbolic geometry interpretation described by the writer at the 1998 PIRT conference. In hyperbolic geometry the sum of the angles of a triangle is less than $\pi$ and correspondingly the rotational angle is found to equal to the hyperbolic deficit D:

$$D = -E = \pi - (A+B+C) \tag{9.2}$$

Reconsider the two examples above.

*EXAMPLE 1:* Two velocities at right angles.

Sommerfeld used spherical trigonometry to derive the formula

$$\sin E = \frac{\sin \varphi_1 . \sin \varphi_2}{1 + \cos \varphi_1 . \cos \varphi_2} \tag{9.3}$$

where $\varphi_1, \varphi_2$ are the Minkowski angles corresponding to the two velocities. Using (2.3), this expression is seen equivalent to the expression (7.6) for sin E from which the formula for the Thomas rotation follows.

Replacing E by - D and $\varphi_1$, $\varphi_2$ by $iw_1$, $iw_2$ we find the formula for the hyperbolic case

$$\sin D = \frac{\mathrm{sh}\, w_1 . \mathrm{sh}\, w_2}{1 + \mathrm{ch}\, w_1 . \mathrm{ch}\, w_2} \tag{9.4}$$

in terms of corresponding rapidities $w_1, w_2$. By (3.2), this is again equivalent to expression (7.6) for sin E.

Sommerfeld only used this method in the case of velocities at right angles although he indicated that it could also be used for inclined velocities. He also mentioned in a footnote the possibility of using hyperbolic geometry.

*EXAMPLE 2* Two velocities inclined at an angle.

Lagrange (1799) gave the following formula for the excess for a spherical triangle where A is one vertex angle and $\beta$, $\gamma$ are the angular lengths of the adjacent sides:

$$\cot(E/2) = \frac{\cos(\beta/2).\cos(\gamma/2) + \sin(\beta/2).\sin(\gamma/2).\cos A}{\sin(\beta/2).\sin(\gamma/2).\sin A} \tag{9.5}$$

On identifying $\beta$, $\gamma$ with $iw_1$, $iw_2$ and putting

$$\theta = \pi - A \qquad (9.6)$$

we find the appropriate formula for deficit D in the hyperbolic representation of velocities $v_1,v_2$ (or the corresponding rapidities $w_1,w_2$) inclined at angle θ

$$\cot(D/2) = \frac{ch(w_1/2)\ ch(w_2/2) + sh(w_1/2)\ sh(w_2/2).\cos\theta}{sh(w_1/2)\ sh(w_2/2).\sin\theta} \qquad (9.7)$$

$$= \frac{\coth(w_1/2)\coth(w_2/2) + \cos\theta}{\sin\theta} \qquad (9.8)$$

from which

$$\tan D/2 = \frac{\sin\theta}{(k + \cos\theta)} \qquad (9.9)$$

where

$$k = \coth(w_1/2)\coth(w_2/2) \qquad (9.10)$$

Now, for any rapidity w

$$\coth^2(w/2) = \frac{ch^2(w/2)}{sh^2(w/2)} = \frac{ch\ w - 1}{ch\ w - 1} \qquad (9.11)$$

so that

$$k^2 = \frac{(ch\ w_1 + 1)\ (ch\ w_2 + 1)}{(ch\ w_1 - 1)\ (ch\ w_2 - 1)} = \frac{(\gamma_1 + 1)(\gamma_2 + 1)}{(\gamma_1 - 1)(\gamma_2 - 1)} \qquad (9.12)$$

giving agreement with the result quoted before. This method does not seem to have appeared in the literature although Smorodinski (1962 etc.) used formulae from which the result could easily have been deduced (see appendix)

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## 11. **Appendix** Smorodinski's Formulae for Hyperbolic Defect.

Smorodinski found expressions for the hyperbolic deficit and hence rotation angle in papers written from 1962, partly reported by Wick (1963) and others. The formulae they quoted use symmetrical expressions from which the formulae used in the text above can be deduced.

In his paper in JETP 1962 Smorodinski quoted as one of the formulae for hyperbolic defect D the expression

$$\cos D/2 = \frac{1 + \mathrm{ch}\,\alpha + \mathrm{ch}\,\beta + \mathrm{ch}\,\gamma}{4(\mathrm{ch}\,\alpha/2)\,(\mathrm{ch}\,\beta/2)\,(\mathrm{ch}\,\gamma/2)} \qquad (11.1)$$

where α, β, γ denote the non-dimensional lengths of the hyperbolic triangle i.e. the lengths divided by the radius of curvature. These correspond to rapidities.

In another paper at the Yerevan Conference 1963 he gave the alternative formula

$$\sin D = \frac{(1 + \mathrm{ch}\,\alpha + \mathrm{ch}\,\beta + \mathrm{ch}\,\gamma).\mathrm{sh}\,\alpha\,\mathrm{sh}\,\beta\,\sin C}{8(\mathrm{ch}^2\,\alpha/2)\,(\mathrm{ch}\,\beta/2)\,(\mathrm{ch}^2\,\gamma/2)} \qquad (11.2)$$

From these two the value of coth D/2 is easily found. Firstly, using

$$\sin D = 2 \sin D/2 \cos D/2 \qquad (11.3)$$

there follows from these two expressions,

$$\sin D/2 = \frac{\mathrm{sh}\,\alpha\,\mathrm{sh}\,\beta\,\sin C\ .}{4(\mathrm{ch}\,\alpha/2)\,(\mathrm{ch}\,\beta/2)\,(\mathrm{ch}\,\gamma/2)} \qquad (11.4)$$

and so by division follows

$$\coth D/2 = \frac{1 + \mathrm{ch}\,\alpha + \mathrm{ch}\,\beta + \mathrm{ch}\,\gamma}{\mathrm{sh}\,\alpha\,\mathrm{sh}\,\beta\,\sin C} \qquad (11.5)$$

The numerator may now be transformed to involve α, β, C:

$$1 + \mathrm{ch}\,\alpha + \mathrm{ch}\,\beta + \mathrm{ch}\,\gamma$$

$$= (\text{ch}\,\alpha - 1)(\text{ch}\,\beta - 1) + \text{ch}\,\gamma - \text{ch}\,\alpha\,\text{ch}\,\beta$$
$$= (\text{ch}\,\alpha + 1)(\text{ch}\,\beta + 1) + \text{sh}\,\alpha\,\text{sh}\,\beta\cos C \qquad (11.6)$$

Then follows

$$\coth D/2 = \frac{(\text{ch}\,\alpha + 1)(\text{ch}\,\beta + 1) + \text{sh}\,\alpha\,\text{sh}\,\beta\cos C}{\text{sh}\,\alpha\,\text{sh}\,\beta\sin C}$$

$$= \frac{\text{ch}\,\alpha/2.\text{ch}\,\beta/2 + \text{sh}\,\alpha/2.\text{sh}\,\beta/2.\cos C}{\text{sh}\,\alpha/2.\text{sh}\,\beta/2.\sin C}$$

$$= \frac{\coth\alpha/2.\coth\beta/2 + \cos C}{\sin C} \qquad (11.7)$$

which is formula (9.8) of the text in a different notation.

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