GAUSSIAN GENERALISATIONS OF THE RELATIVITY THEORY FUNDAMENTS WITH APPLICATIONS

Lyubomir T. Gruyitch – University of Technology Belfort–Montbeliard (UTBM),
Site de Belfort, Rue Thierry Mieg - 90010 Belfort Cedex - France

1. INTRODUCTION

Newton [23] himself had recognised the importance of time scales and units. Different physical sub-processes of a complex physical process can have different natural speeds of their evolutions [9, 21, 22, 24 - 28]. This justifies the use of multiple time scales.

Newton [23] explained the meaning and defined the sense of both the absolute time and *the relative time.* They have been recently reproved correct in [7, 8, 11, 12], where it has been shown that Einstein's meaning of time relativity is just that what Newton had explained as time relativity.

A crucial point of Einstein's relativity theory is the postulate on the constancy of the velocity of the light in vacuum relative to inertial frames. Another inherent point of Einstein's general relativity theory is the invariance of the time-space length under the Gaussian transformation [2-6]. The Lorentz-Einstein relativity theory is based on and requires equality of time scaling coefficients and of those of space scaling in the Lorentz transformation. Besides, their values are assumed constant [2-16, 14-18].

By relying on the nature of *time*, as explained by ***Newton*** [23], it was shown in [7, 8, 11, 12] that the time scaling coefficients should be different, as well as those of space scaling, which led to new results on time and space co-ordinate transformations. They incorporate those by Lorentz as a singular case. As a consequence, new formulas were proved for the basic physical variables: speed, acceleration, force and energy, as well as for mass. They resulted also from the Euclidean transformation of the time-space length.

In reality, we use different time units and scales at different places. Hence, it is reasonable to allow the scaling factors to depend on the co-ordinates of a position of the place where the measurement takes place. This will be permitted in what follows, together with different time scaling (coefficient) functions and different space scaling (coefficient) functions. Furthermore, the Gaussian transformation will be applied to time-space length transformation rather than Euclidean. We will examine implications on the basic physical variables, by establishing new formulas for speed, acceleration, force, energy, as well as for mass.

The properties of *time* combined with a new physical principle – the Physical Continuity and Uniqueness Principle [10] – enabled a development of novel directions in the control science and engineering. They will be broadened to the synthesis of the natural tracking control of mechanical systems (aeroplanes, missiles, rockets, ships and vehicles).

2. TIME FEATURES

It was shown that the following Newtonian explanation of the crucial features of *time* is correct [7, 8, 11, 12]:

Time *is* ***an independent physical variable.*** *Its value is strictly monotonously continuously increasing independently of all other (physical and mathematical) variables, processes and events*.

The ***value of time*** is called **moment** or **instant.** It is denoted by t and a subscript, e.g. t_2. An arbitrary instant will be denoted as time itself by t. The time value is determined accurately up to an unknown additive constant. *A* ***time values sequence*** *uniquely determines the order of events happening.*

Moment (***instant***) reflects *an instantaneous internal physical situation* of a material object called its **age**. **Time value difference** (**time interval**) is used to ***measure duration*** *of a process, of a movement, of a rest or of the existence of an object*.

Different materials can have *different speeds of evolution of physical processes*, which can hold also for the same material object at its different points. For this reason, different time scales (e.g. decimal, logarithmic), different initial moments and/or different time units (e.g. second, minute) can be assigned to different material objects and/or to different parts of the same material object giving a relative meaning to *time* in this sense. This is *Newtonian meaning of the relativity of time*. It should be emphasised that **Newton** himself [23: pp. 8 – 10] introduced and explained both the absolute and relative sense of *time*. Newton's explanation of the relativity of *time* incorporates that of Einstein [2, p. 20], [4, pp. 26 – 27], [5, pp. 23 – 40], which was explained and proved in [7, 8, 11, 12].

A co-ordinate axis used for a time axis is immovable relative to the environment. It is denoted by *T*. It is an ordered set of all instants: T ={t: t∈R,

$t \in C^{(1)}(R)$, dt > 0}. Once a time axis has been accepted with a fixed time scale including a fixed time interval unit, then the zero moment should have been also accepted. Afterwards, we can select any instant $t_0 \in T$ for an initial instant. We accept that it has been chosen and fixed. It can be $t_0 = 0$, but need not.

3. TIME, SPEED AND COORDINATE TRANSFORMATION

What is presented in this section represents the new relativity theory fundaments. They lead to new results on the basic physical variables and on mass, which are established in the next sections.

3.1. *Time scale and speed*

A time scale of a time axis *T* can be variously accepted. Different time scales are associated with *T*: "an original time scale" *T* that is not indexed and "i"-time scale T_i. A time value (instant) measured in *T*-scale and T_i-scale is designated by t and t_i, respectively, $t \in T$ and $t_i \in T_i$. We usually accept various time scales at different locations. Therefore, time scaling (coefficient) functions $\mu_i(.)$: $R^n \rightarrow R^+$, $R^+ =]0, \infty[$, depend in general on the position vector $\mathbf{r}_i \in R_i^n$:

$$t_i - t_{i0} = \mu_i(\mathbf{r}_i)(t - t_0),\ t_{i0} = \mu_i(\mathbf{r}_i)t_0,\ i = 1, 2, \ldots, s. \quad (1)$$

The time scale factor functions $\mu_i(.)$ are positive valued. Their values depend on the instantaneous position vector $\mathbf{r}_i$ of a point at which time value is measured. Immovable time axes (such as *T* and T_i) are only considered in what follows. The Cartesian product set TxR^n is denoted by *J*, $J = T\text{x}R^n$. It is the *(n+1)-dimensional real integral vector space*, for short *the integral space* [11, 12]. A pair (t, x) is called *an event in J* [13]. It can happen only once due to the properties of *time*. The integral space corresponding to the time axis T_i is J_i: $J_i = T_i \text{ x } R_i^n$, i = 1, 2, …, s.

A path (vector) $d\mathbf{r}_L$ passed by a light array over an infinitesimal time interval dt is the light speed (vector) $\mathbf{c}(t) = d\mathbf{r}_L(t)/dt$ at moment t. The value of the light speed $c(t) = ||\mathbf{c}(t)||$ is constant in vacuum: $c(t) \equiv c$ [3: p. 15, 4: p. 26]. It is the light speed with respect to the vacuum and measured relative to *T*, for short: the light speed. The norm || **.** || is any accepted norm on R^n. Besides, c^i is the value of the light speed measured with respect to R_i^n and relative to $t_i \in T_i$. In what follows, the environment will be arbitrary but fixed and such that it enables a constant light speed. The environment dimension n ∈{1, 2, ... }. It is an n-dimensional real vector space R^n. A fixed point O is accepted in the environment R^n to be the origin of a system of reference (a co-ordinate system). We accept another point O_i, possibly movable relative to the environment, to be the origin of the co-ordinate system R_i^n. The speed $\mathbf{v}_{O_i}(t)$ of the point O_i and of the reference system R_i^n relative to O and R^n is accepted constant: $\mathbf{v}_{O_i}(t) \equiv \mathbf{v}_{O_i} = v_{O_i}\mathbf{r}_0$, where $\mathbf{r}_0$ is a unity vector, $\| \mathbf{r}_0 \| = 1$. The unity vector $\mathbf{r}_0 \in R^n$ is also used to represent symbolically the spaces R^n and R_i^n, i = 1, 2, …, s.

Without losing in generality, we represent the position of an arbitrary and fixed, possibly movable, point A in the spaces R^n and R_i^n by vectors $\mathbf{r}_A^O = \mathbf{r} \in R^n$, $\mathbf{r} = \rho\mathbf{r}_0$, and $\mathbf{r}_A^{O_i} = \mathbf{r}_i \in R_i^n$, $\mathbf{r}_i = \rho_i\mathbf{r}_0$, i = 1, 2, …, s, respectively. Their lengths are expressed by their norms $\rho = ||\mathbf{r}||$ and $\rho_j = ||\mathbf{r}_j||$, respectively. Velocities should be naturally permitted to depend on a time scale used:

$\mathbf{v}_A^{O_i}(t_i; t_{i0}) \equiv v_A^{O_i}(t_i; t_{i0})\mathbf{r}_0$ is the speed of the point A relative to O_i measured in terms of t_i; if and only if it is constant then it is denoted by $\mathbf{v}_A^{O_i} = v_A^{O_i}\mathbf{r}_0 = \mathbf{const.}$, i = 1, 2, … , s,

$\mathbf{v}_{O_k}^m \equiv v_{O_k}^m \mathbf{r}_0$ is the constant speed of O_k relative to O measured in terms of t_m, k, m = 1, 2, …, s,

$v_{O_i}^m \le v_{O_j}^m$ is accepted, m∈{i, j},

$\boldsymbol{v}_{ji}^k \equiv v_{ji}^k \mathbf{r}_0 \equiv (v_{O_j}^k - v_{O_i}^k)\mathbf{r}_0$ is the constant relative speed of O_j with respect to O_i measured all in terms of t_κ, k ∈ {i, j}. It is to be noted that $v_{ji}^j > -c^j$ due to the accepted $v_{O_i}^j \le v_{O_j}^j$ and due to $c^j > 0$.

If the point A represents a light signal, then:

$\mathbf{r}_A^O = \mathbf{r} = \mathbf{r}_L \in R^n$, $\mathbf{r}_L = \rho_L\mathbf{r}_0 = \rho\mathbf{r}_0$, $\mathbf{r}_A^{O_i} = \mathbf{r}_i = \mathbf{r}_{Li} \in R_i^n$, $\mathbf{r}_{Li} = \rho_{Li}\mathbf{r}_0 = \rho_i\mathbf{r}_0$ and $\mathbf{v}_A^{O_i} = \mathbf{v}_L^{O_i} = \mathbf{c}^i = c^i\mathbf{r}_0$.

3.2. Problem Statement

What are conditions on the scaling (coefficient) functions $\mu_i(.): R_i^n \rightarrow R^+$, $\alpha_i^j(.): R_i^n \text{ x } R_j^n \rightarrow R^+$, $\alpha_j^i(.): R_i^n \text{ x } R_j^n \rightarrow R^+$, $k_i^j(.): R_i^n \text{ x } R_j^n \rightarrow R^+$ and $k_j^i(.): R_i^n \text{ x } R_j^n \rightarrow R^+$, so that the equations (2), (3),

$$t_i - t_{i0} = \alpha_j^i(\mathbf{r}_i, \mathbf{r}_j)[(t_j - t_{j0}) + \frac{v_{ji}^j}{u^j w^j}\rho_j(t_j; t_{j0})], \quad (2a)$$

$$t_j - t_{j0} = \alpha_i^j(\mathbf{r}_i,\mathbf{r}_j)[(t_i - t_{i0}) - \frac{v_{ji}^i}{u^i w^i}\rho_i(t_i;t_{i0})], \qquad (2b)$$

$$\mathbf{r}_i(t_i;t_{i0}) = k_j^i(\mathbf{r}_i,\mathbf{r}_j)[\mathbf{r}_j(t_j;t_{j0}) + \boldsymbol{v}_{ji}^j(t_j - t_{j0})], \qquad (3a)$$

$$\mathbf{r}_j(t_j;t_{j0}) = k_i^j(\mathbf{r}_i,\mathbf{r}_j)[\mathbf{r}_i(t_i;t_{i0}) - \boldsymbol{v}_{ji}^i(t_i - t_{i0})], \qquad (3b)$$

$$u^{(.)}, w^{(.)} \in \left\{c^{(.)}, v_A^{O_{(.)}}(t_{(.)};t_{(.)0}), v_A^{O_{(.)}}\right\}, \qquad (3c)$$

$$\mathbf{r}_{(.)} = \mathbf{r}_{(.)}(t_{(.)};t_{(.)0}), (.)\in\{i, j\}, \qquad (3d)$$

hold and that they imply the identity (4),

$$\begin{bmatrix} \mathbf{r}_i^T & (t_i - t_{i0})\left[\mathbf{v}_A^{O_i}(t_i;t_{i0})\right]^T \end{bmatrix} G \bullet \begin{bmatrix} \mathbf{r}_i^T & (t_i - t_{i0})\left[\mathbf{v}_A^{O_i}(t_i;t_{i0})\right]^T \end{bmatrix}^T \equiv \begin{bmatrix} \mathbf{r}_j^T & (t_j - t_{j0})\left[\mathbf{v}_A^{O_j}(t_j;t_{j0})\right]^T \end{bmatrix} G \bullet \begin{bmatrix} \mathbf{r}_j^T & (t_j - t_{j0})\left[\mathbf{v}_A^{O_j}(t_j;t_{j0})\right]^T \end{bmatrix}^T \; ? \qquad (4)$$

The matrix G = blockdiag{A -B}, A and B are positive definite matrices with A = B possible but not required, $A \text{ and } B \in R^{n x n}$.

The time and space co-ordinate transformations (2) and (3) permit in general both different scaling coefficient functions and their dependence on a position and choice of an arbitrary point A, which can be movable. In such a case, the transformations are non-uniform and they are the starting basis for a new direction in the relativity theory –*the strict non-uniform relativity theory*. If we impose a condition that the scaling coefficient functions and the Gaussian transformation (4) should be independent of a position and of a choice of an arbitrary point A then the transformations are non-uniform and they set a basis for *the non-uniform relativity theory*.

The time-scale equations (1) are inherent for multiple time scale dynamical systems [7, 8, 11, 12]. The condition (4) expresses the Gaussian transformation of the time-space length. It is a crucial condition of Einstein's general relativity theory for validity of the co-ordinate transformation defined by (2) and (3) [2, 3]. If G = I is the identity matrix, then the transformation becomes Euclidean, which is used in the special relativity theory.

3.3. *Problem Solution*

Two cases should be distinguished and will be considered relative to the mutual relationship among scaling coefficient functions, which can be constant in either case:

General case: $\alpha_i^j(.) \neq \alpha_j^i(.)$ and/or $k_i^j(.) \neq k_j^i(.)$.

Special case: $\alpha_i^j(.) = \alpha_j^i(.)$ and $k_i^j(.) = k_j^i(.)$.

Problem Solutions for the General Case

Theorem 1. If the speed of an accepted arbitrary point A is constant then, in order for the scaling coefficient functions $\alpha_i^j(.)$, $\alpha_j^i(.)$, $\alpha_i^j(.) \neq \alpha_j^i(.)$, $k_i^j(.)$ and $k_j^i(.)$, $k_i^j(.) \neq k_j^i(.)$, to obey (2) and (3), and for (1) through (3) to imply (4), it is necessary and sufficient that they are constant and that the following equations hold for any choice of $\mu_i(.): R^n \to R^+$:

$$\alpha_j^i(\mathbf{r}_i,\mathbf{r}_j) = \frac{\mu_i(\mathbf{r}_i)}{\mu_j(\mathbf{r}_j)}\frac{1}{1+\frac{v_{ji}^j v_A^{O_j}}{u^j w^j}} = \frac{v_A^{O_j}}{v_A^{O_i}}\frac{1}{1+\frac{v_{ji}^j v_A^{O_j}}{u^j w^j}}, \qquad (5a)$$

$$u^j, w^j \in \left\{c^j, v_A^{O_j}\right\},$$

$$\alpha_i^j(\mathbf{r}_i,\mathbf{r}_j) = \frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)}\frac{1}{1-\frac{v_{ji}^i v_A^{O_i}}{u^i w^i}} = \frac{v_A^{O_i}}{v_A^{O_j}}\frac{1}{1-\frac{v_{ji}^i v_A^{O_i}}{u^i w^i}}, \qquad (5b)$$

$$u^i, w^i \in \left\{c^i, v_A^{O_i}\right\},$$

$$v_{ji}^i < \frac{u^i w^i}{v_A^{O_i}}, \; u^i, w^i \in \left\{c^i, v_A^{O_i}\right\}, \qquad (6)$$

$$k_j^i(\mathbf{r}_i,\mathbf{r}_j) = \frac{1}{1+\frac{v_{ji}^j}{v_A^{O_j}}} \equiv k_j^i = \text{const.}, \qquad (7a)$$

$$k_i^j(\mathbf{r}_i,\mathbf{r}_j) = \frac{1}{1-\frac{v_{ji}^i}{v_A^{O_i}}} \equiv k_i^j = \text{const.}, \qquad (7b)$$

and

$$\frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} = \frac{v_A^{O_i}}{v_A^{O_j}} = \text{const.} \qquad (8)$$

Proof. It is omitted due to the space limitation.

The equations (5) and (7) give the next form to the equations (2):

$$t_i - t_{i0} = \frac{\mu_i(\mathbf{r}_i)}{\mu_j(\mathbf{r}_j)} \frac{(t_j - t_{j0}) + \frac{v_{ji}^j}{u^j w^j}\rho_j(t_j; t_{j0})}{1 + \frac{v_{ji}^j v_A^{O_j}}{u^j w^j}} =$$

$$\frac{v_A^{O_j}}{v_A^{O_i}} \frac{(t_j - t_{j0}) + \frac{v_{ji}^j}{u^j w^j}\rho_j(t_j; t_{j0})}{1 + \frac{v_{ji}^j v_A^{O_j}}{u^j w^j}}, \qquad (2a')$$

$$t_j - t_{j0} = \frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} \frac{(t_i - t_{i0}) - \frac{v_{ji}^i}{u^i w^i}\rho_i(t_i; t_{i0})}{1 - \frac{v_{ji}^i v_A^{O_i}}{u^i w^i}} =$$

$$\frac{v_A^{O_i}}{v_A^{O_j}} \frac{(t_i - t_{i0}) - \frac{v_{ji}^i}{u^i w^i}\rho_i(t_i; t_{i0})}{1 - \frac{v_{ji}^i v_A^{O_i}}{u^i w^i}}, \qquad (2b')$$

and to the equations (3),

$$\mathbf{r}_i(t_i; t_{i0}) = \frac{\mathbf{r}_j(t_j; t_{j0}) + \boldsymbol{v}_{ji}^j(t_j - t_{j0})}{1 + \frac{v_{ji}^j}{v_A^{O_j}(\mathbf{r}_j)}}, \qquad (3a')$$

$$\mathbf{r}_j(t_j; t_{j0}) = \frac{\mathbf{r}_i(t_i; t_{i0}) - \boldsymbol{v}_{ji}^i(t_i - t_{i0})}{1 - \frac{v_{ji}^i}{v_A^{O_i}(\mathbf{r}_i)}}. \qquad (3b')$$

Theorem 1 establishes the basic result for the strict non-uniform relativity theory. The time scaling factors can take one of four different forms determined by a choice of $u^{(.)}$ and $w^{(.)}$ in $\left\{c^{(.)}, v_A^{O_{(.)}}\right\}$, $(.)\in\{i,\ j\}$. Hence, there are four different forms of the co-ordinate transformations (2) and (3), equivalently, (2') and (3').

Theorem 2. If the scaling factors of the transformations (2) through (4) and the transformations themselves should be independent of a choice of an arbitrary point A then they become (9) through (11):

$$t_i - t_{i0} = \alpha_j^i[(t_j - t_{j0}) + \frac{v_{ji}^j}{(c^j)^2}\rho_j(t_j; t_{j0})], \qquad (9a)$$

$$t_j - t_{j0} = \alpha_i^j[(t_i - t_{i0}) - \frac{v_{ji}^i}{(c^i)^2}\rho_i(t_i; t_{i0})], \qquad (9b)$$

$$\mathbf{r}_i(t_i; t_{i0}) = k_j^i[\mathbf{r}_j(t_j; t_{j0}) + \boldsymbol{v}_{ji}^j(t_j - t_{j0})], \qquad (10a)$$

$$\mathbf{r}_j(t_j; t_{j0}) = k_i^j[\mathbf{r}_i(t_i; t_{i0}) - \boldsymbol{v}_{ji}^i(t_i - t_{i0})], \qquad (10b)$$

$$\left[\mathbf{r}_i^T(t_i; t_{i0}) \ \ (t_i - t_{i0})\mathbf{c}^{i^T}\right] G \left[\mathbf{r}_i^T(t_i; t_{i0}) \ \ (t_i - t_{i0})\mathbf{c}^{i^T}\right]^T \equiv$$

$$\left[\mathbf{r}_j^T(t_j; t_{j0}) \ \ (t_j - t_{j0})\mathbf{c}^{j^T}\right] G \left[\mathbf{r}_j^T(t_j; t_{j0}) \ \ (t_j - t_{j0})\mathbf{c}^{j^T}\right]^T. \qquad (11)$$

Theorem 3. In order for the scaling coefficients $\alpha_i^j \in R^+$, $\alpha_j^i \in R^+$, $\alpha_i^j \neq \alpha_j^i$, $k_i^j \in R^+$ and $k_j^i \in R^+$, $k_i^j \neq k_j^i$, to obey (9) and (10), and for (1), (9) and (10) to imply (11) it is necessary and sufficient that the following equations hold for any choice of $\mu_i(.): R_i^n \to R^+$:

$$\alpha_j^i = \frac{\mu_i(\mathbf{r}_i)}{\mu_j(\mathbf{r}_j)} \frac{1}{1 + \frac{v_{ji}^j}{c^j}} = \frac{c^j}{c^i} \frac{1}{1 + \frac{v_{ji}^j}{c^j}}, \qquad (12a)$$

$$\alpha_i^j = \frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} \frac{1}{1 - \frac{v_{ji}^i}{c^i}} = \frac{c^i}{c^j} \frac{1}{1 - \frac{v_{ji}^i}{c^i}}, \qquad (12b)$$

$$v_{ji}^i < c^i, \qquad (13)$$

$$k_j^i = \frac{1}{1 + \frac{v_{ji}^j}{c^j}}, \qquad (14a)$$

$$k_i^j = \frac{1}{1 - \frac{v_{ji}^i}{c^i}}, \qquad (14b)$$

and

$$\frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} = \frac{c^i}{c^j} = \text{const.} \qquad (15)$$

Proof. It is omitted due to the space limitation.

In the case the co-ordinate systems R_i^n and R_j^n move with the same velocities then the next theorem is easily proved by setting $v_{ji}^i = v_{ji}^j = 0$ in the Theorem 1:

Theorem 4. If the speed of an arbitrary point A is constant and if the co-ordinate systems R_i^n and R_j^n move with the same velocities then, in order for the scaling coefficients α_i^j, α_j^i, $\alpha_i^j \neq \alpha_j^i$, k_i^j and k_j^i, $k_i^j \neq k_j^i$, to obey (2) and (3), and for (1) through (3) to imply (4), it is necessary and sufficient that they are constant and that the following equations hold for any choice of $\mu_i(.)$: $R_i^n \to R^+$:

$$\alpha_j^i = \frac{\mu_i(\mathbf{r})}{\mu_j(\mathbf{r})} = \frac{v_A^{Oj}}{v_A^{O_i}} = \frac{c^j}{c^i},\ \alpha_i^j = \frac{\mu_j(\mathbf{r})}{\mu_i(\mathbf{r})} = \frac{v_A^{O_i}}{v_A^{O_j}} = \frac{c^i}{c^j},\quad (16)$$

$$k_j^i = k_i^j = 1. \quad (17)$$

This result shows that different time scales can be chosen in the case $\alpha_i^j \neq \alpha_j^i$ even the co-ordinate systems R_i^n and R_j^n move with the same velocities. This is a consequence of the time independence of the space. Then the time scale coefficients are related to the values of the light speed by the equation (16). This is useful to know for dynamical systems described by the following non-linear vector differential equations:

$$\frac{dx}{dt} = f(t,x,y,M), x \in R^n, y \in R^s, M = \operatorname{diag}\{\mu_1 \dots \mu_s\},$$
$$M\frac{dy}{dt} = g(t,x,y,M).$$

Problem Solutions for the Special Case

Theorem 5. In order for the scaling coefficients α_i^j, α_j^i, $\alpha_i^j = \alpha_j^i = \alpha \in R^+$, k_i^j and k_j^i, $k_i^j = k_j^i = k \in R^+$, to obey (9) and (10), and for (1), (9) and (10) to imply (11) it is necessary and sufficient that the following equations hold for any choice of $\mu_i(.): R_i^n \to R^+$:

$$c^i = c^j = c_{ij} = c_{ji}, \quad (18)$$

$$v_{ij}^i = v_{ij}^j = v_{ij} = -v_{ji}, \quad (19)$$

$$\alpha = k = \left(1 - \frac{v_{ij}^2}{c_{ij}^2}\right)^{-1/2}, \quad (20)$$

$$\mu_j(\mathbf{r}) = \mu_i(\mathbf{r})\left(1 - \frac{v_{ij}}{c_{ij}}\right)\left(1 + \frac{v_{ij}}{c_{ij}}\right)^{-1/2}. \quad (21)$$

Proof. It is omitted due to the space limitation.

The preceding theorem gives the following special form to the equations (9):

$$t_i - t_{i0} = \frac{(t_j - t_{j0}) - \frac{v_{ij}}{c_{ij}}\rho_j(t_j; t_{j0})}{\sqrt{1 - \frac{v_{ij}^2}{c_{ij}^2}}}, \quad (22a)$$

$$t_j - t_{j0} = \frac{(t_i - t_{i0}) + \frac{v_{ij}}{c_{ij}}\rho_i(t_i; t_{i0})}{\sqrt{1 - \frac{v_{ij}^2}{c_{ij}^2}}}, \quad (22b)$$

and to the equations (10),

$$\mathbf{r}_i(t_i; t_{i0}) = \frac{\mathbf{r}_j(t_j; t_{j0}) - \boldsymbol{v}_{ij}(t_j - t_{j0})}{\sqrt{1 - \frac{v_{ij}^2}{c_{ij}^2}}}, \quad (23a)$$

$$\mathbf{r}_j(t_j; t_{j0}) = \frac{\mathbf{r}_i(t_i; t_{i0}) + \boldsymbol{v}_{ij}(t_i - t_{i0})}{\sqrt{1 - \frac{v_{ij}^2}{c_{ij}^2}}}. \quad (23b)$$

These equations incorporate the well known equations of the Lorentz transformation [14-19], which are basic for the Lorentz - Einstein theory of relativity [2-6, 14-19]. It is now obvious that they appear a singular case of the transformations (9) and (10). It should be also noted that they follow herein from the accepted time explanation in Newton's sense.

4. DYNAMICAL VARIABLES AND CO-ORDINATE TRANSFORMATION

In this and next sections new relationships will be derived by employing the generalised Lorentz transformation (2), (3), or (9), (10).

4.1 Speed and co-ordinate transformation

The general case

Theorem 6. If the speed of an arbitrary point A is constant: $\mathbf{v}_A^{O_i}(t_i; t_{i0}) = \mathbf{v}_A^{O_i}$, if the scaling coefficient functions $\alpha_i^j(.) \in R^+$, $\alpha_j^i(.) \in R^+$, $\alpha_i^j(.) \neq \alpha_j^i(.)$, and $k_i^j(.)$, $k_j^i(.) \in R^+$, $k_i^j(.) \neq k_j^i(.)$, obey (2) and (3), and if (1) through (3) imply (4), then the speed $\mathbf{v}_A^{O_i}$ of the point A with respect to the origin O_i of R_i^n expressed in terms of t_i and the speed $\mathbf{v}_A^{O_j}$ of the same point A with respect to the origin O_j of R_j^n expressed in terms of t_j are interrelated as follows:

$$\mathbf{v}_A^{O_i} = \frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} \mathbf{v}_A^{O_j} = \frac{c^i}{c^j} \mathbf{v}_A^{O_j}. \tag{24}$$

<u>Proof.</u> It is omitted due to the space limitation.

Theorem 7. Let the scaling coefficients $\alpha_i^j \in R^+$ and $\alpha_j^i \in R^+$, $\alpha_i^j \neq \alpha_j^i$, $k_i^j \in R^+$, $k_j^i \in R^+$, $k_i^j \neq k_j^i$, obey (9) and (10), and let (1), (9) and (10) imply (11). Then the speed $\mathbf{v}_A^{O_i}(t_i;t_{i0})$ of a point A with respect to the origin O_i of R_i^n expressed in terms of t_i and the speed $\mathbf{v}_A^{O_j}(t_j;t_{j0})$ of the same point A with respect to the origin O_j of R_j^n expressed in terms of t_j are interrelated as follows:

$$\mathbf{v}_A^{O_i}(t_i;t_{i0}) = \frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} \frac{\mathbf{v}_A^{O_j}(t_j;t_{j0}) + \boldsymbol{v}_{ji}^j}{1 + \frac{v_{ji}^j \mathrm{v}_A^{O_j}(t_j;t_{j0})}{(c^j)^2}} = \frac{c^i}{c^j} \frac{\mathbf{v}_A^{O_j}(t_j;t_{j0}) + \boldsymbol{v}_{ji}^j}{1 + \frac{v_{ji}^j \mathrm{v}_A^{O_j}(t_j;t_{j0})}{(c^j)^2}}, \tag{25a}$$

and

$$\mathbf{v}_A^{O_j}(t_j;t_{j0}) = \frac{\mu_i(\mathbf{r}_i)}{\mu_j(\mathbf{r}_j)} \frac{\mathbf{v}_A^{O_i}(t_i;t_{i0}) - \boldsymbol{v}_{ji}^j}{1 - \frac{v_{ji}^j \mathrm{v}_A^{O_i}(t_i;t_{i0})}{(c^j)^2}} = \frac{c^j}{c^i} \frac{\mathbf{v}_A^{O_i}(t_i;t_{i0}) - \boldsymbol{v}_{ji}^j}{1 - \frac{v_{ji}^j \mathrm{v}_A^{O_i}(t_i;t_{i0})}{(c^j)^2}}. \tag{25b}$$

The velocity transformation equations (25) are beyond the Einstein relativity theory. The former are general, while the latter result from the special case. Notice that $v^i = v^j = v_{ij}$ and $c^i = c^j = c_{ij}$ in the special case.

It should be also emphasised that <u>Theorem 6 and Theorem 7 result from the Newtonian explanation of the features of *time*.</u>

The special case

Theorem 8. Let the scaling coefficients $\alpha_i^j \in R^+$ and $\alpha_j^i \in R^+$, $\alpha_i^j = \alpha_j^i$, $k_i^j \in R^+$, $k_j^i \in R^+$, $k_i^j = k_j^i$, obey (9) and (10), and let (1), (9) and (10) imply (11). Then the speed $\mathbf{v}_A^{O_i}(t_i;t_{i0})$ of a point A with respect to the origin O_i of R_i^n expressed in terms of t_i and the speed $\mathbf{v}_A^{O_j}(t_j;t_{j0})$ of the same point A with respect to the origin O_j of R_j^n expressed in terms of t_j are interrelated as follows:

$$\mathbf{v}_A^{O_i}(t_i;t_{i0}) = \frac{\mu_j(\mathbf{r}_j)}{\mu_i(\mathbf{r}_i)} \frac{\mathbf{v}_A^{O_j}(t_j;t_{j0}) - \boldsymbol{v}_{ij}}{1 - \frac{v_{ij} \mathrm{v}_A^{O_j}(t_j;t_{j0})}{(c_{ij})^2}} = \frac{c^i}{c^j} \frac{\mathbf{v}_A^{O_j}(t_j;t_{j0}) - \boldsymbol{v}_{ij}}{1 - \frac{v_{ij} \mathrm{v}_A^{O_j}(t_j;t_{j0})}{(c_{ij})^2}}, \tag{26a}$$

and

$$\mathbf{v}_A^{O_j}(t_j;t_{j0}) = \frac{\mu_i(\mathbf{r}_i)}{\mu_j(\mathbf{r}_j)} \frac{\mathbf{v}_A^{O_i}(t_i;t_{i0}) + \boldsymbol{v}_{ij}}{1 + \frac{v_{ij} \mathrm{v}_A^{O_i}(t_i;t_{i0})}{(c_{ij})^2}} = \frac{c^j}{c^i} \frac{\mathbf{v}_A^{O_i}(t_i;t_{i0}) + \boldsymbol{v}_{ij}}{1 + \frac{v_{ij} \mathrm{v}_A^{O_i}(t_i;t_{i0})}{(c_{ij})^2}}. \tag{26b}$$

Notice that the equations (26) can be also derived directly from the equations (25) for $c^i = c^j = c_{ij} = c_{ji}$.

The equations (26) are well known in the relativity theory. It is important to note <u>that they are derived herein starting with the explanation of time properties in the Newtonian sense</u>.

4.2. Acceleration and co-ordinate transformation

The general case

Theorem 9. Let the scaling coefficients $\alpha_i^j \in R^+$ and $\alpha_j^i \in R^+$, $\alpha_i^j \neq \alpha_j^i$, $k_i^j \in R^+$, $k_j^i \in R^+$, $k_i^j \neq k_j^i$, obey (9) and (10), and let (1), (9) and (10) imply (11). Then the acceleration $\mathbf{a}_A^{O_i}(t_i;t_{i0})$ of a point A with respect to the origin O_i of R_i^n expressed in terms of t_i and the acceleration $\mathbf{a}_A^{O_j}(t_j;t_{j0})$ of the same point A with respect to the origin O_j of R_j^n expressed in terms of t_j are interrelated as follows:

$$\mathbf{a}_A^{O_i}(t_i;t_{i0}) = \left(\frac{c^i}{c^j}\right)^2\left(1+\frac{v_{ij}^j}{c^j}\right)\frac{\left(1-\frac{v_{ij}^j}{c^j}\right)^2}{\left[1-\frac{v_{ij}^j v_A^{O_j}(t_j;t_{j0})}{(c^j)^2}\right]^3}\mathbf{a}_A^{O_j}(t_j;t_{j0}), \tag{27a}$$

and

$$\mathbf{a}_A^{O_j}(t_j;t_{j0}) = \left(\frac{c^j}{c^i}\right)^2\left(1-\frac{v_{ij}^i}{c^i}\right)\frac{\left(1+\frac{v_{ij}^i}{c^i}\right)^2}{\left[1+\frac{v_{ij}^i v_A^{O_i}(t_i;t_{i0})}{(c^i)^2}\right]^3}\mathbf{a}_A^{O_i}(t_i;t_{i0}). \tag{27b}$$

Theorem 9 results from the Newtonian explanation of the features of *time*.

The acceleration transformation equations (27) are beyond those of the relativity theory. The former are general, while the latter result from the special case as shown in what follows.

The special case

Theorem 10. Let the scaling coefficients $\alpha_i^j \in R^+$ and $\alpha_j^i \in R^+$, $\alpha_i^j = \alpha_j^i$, $k_i^j \in R^+$, $k_j^i \in R^+$, $k_i^j = k_j^i$, obey (9) and (10), and let (1), (9) and (10) imply (11). Then the acceleration $\mathbf{a}_A^{O_i}(t_i;t_{i0})$ of a point A with respect to the origin O_i of R_i^n expressed in terms of t_i and the acceleration $\mathbf{a}_A^{O_j}(t_j;t_{j0})$ of the same point A with respect to the origin O_j of R_j^n expressed in terms of t_j are interrelated as follows:

$$\mathbf{a}_A^{O_i}(t_i;t_{i0}) = \frac{\left[1-\left(\frac{v_{ij}}{c_{ij}}\right)^2\right]^{3/2}}{\left[1-\frac{v_{ij} v_A^{O_j}(t_j;t_{j0})}{c_{ij}^2}\right]^3}\mathbf{a}_A^{O_j}(t_j;t_{j0}), \tag{28a}$$

and

$$\mathbf{a}_A^{O_j}(t_j;t_{j0}) = \frac{\left[1-\left(\frac{v_{ij}}{c_{ij}}\right)^2\right]^{3/2}}{\left[1+\frac{v_{ij} v_A^{O_i}(t_i;t_{i0})}{c_{ij}^2}\right]^3}\mathbf{a}_A^{O_i}(t_i;t_{i0}). \tag{28b}$$

Notice that the equations (28) cannot be derived directly from the equations (27) for $c^i = c^j = c_{ij}$. The equations (28) incorporate those well known in the relativity theory. They are derived herein by starting with the time explanation in the Newtonian sense.

5. MASS AND CO-ORDINATE TRANSFORMATION

Let the example of two particles of equal masses at the rest considered by D'Inverno [1; pp. 45-47] be reconsidered. They are assumed to collide inelastically.

The three inertial co-ordinate systems (reference frames) are accepted. The co-ordinate system R^n is the reference co-ordinate system for other two co-ordinate systems R_i^n and R_j^n.

The frame R_i^n is the "centre-of-mass" frame. Both particles move relative to the frame R_i^n. Their velocities have the same absolute value U^i measured in time scale T_i and before the collision, but opposite sense. Their common speed after the collision equals zero relative to the reference frame R_i^n.

The reference frame R_j^n is tied with the particle No. 2 before the collision. The second particle 2 is evidently at the rest relative to R_j^n before the collision. Then, they do not move relative to R^n.

Both particles move evidently with the common speed U^i with respect to the frame R_j^n after the collision, when the speed is measured relative to T_i. Their common speed value is U^j with respect to the frame R_j^n after the collision if the speed is measured relative to T_j. Hence, (16), as well as (24), provides the following relationship:

$$\frac{U^i}{U^j} = \frac{\mu_i(\mathbf{r}_i)}{\mu_j(\mathbf{r}_j)} = \frac{c^i}{c^j}. \tag{29}$$

The general case

Theorem 11. Let two identical particles 1 and 2 have the same mass m_0 at the rest relative to a reference integral space TxR^n. Let another two integral spaces T_ixR_i^n and T_jxR_j^n be accepted, $T_i \neq T_j$. Let the space co-ordinate system R_i^n be the "centre-of-mass" frame for the particles 1 and 2. Let the space co-ordinate system R_j^n be at the rest relative to R^n. Then, the mass $m\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right]$ of the particle 1 moving with the speed $\mathbf{v}_1^{O_j}(t_j;t_{j0}) = v_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ is related to its mass m_0 at the rest relative to R_j^n by:

$$m\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right] = \frac{m_0}{\sqrt{1-\left(\dfrac{v_1^{O_j}(t_j;t_{j0})}{c^j}\right)^2}}. \tag{30}$$

This generalises the expression for the relativistic mass that exists in the relativity theory and which can be determined now easily for the special case.

The special case

Theorem 12. Let two identical particles 1 and 2 have the same mass m_0 at the rest relative to a reference integral space TxR^n. Let another two integral spaces T_ixR_i^n and T_jxR_j^n be accepted, $T_i = T_j$. Let the space co-ordinate system R_i^n be the "centre-of-mass" frame for the particles 1 and 2. Let the space co-ordinate system R_j^n be at the rest relative to R^n. Then, the mass $m\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right]$ of the particle 1 moving with the speed $\mathbf{v}_1^{O_j}(t_j;t_{j0}) = v_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ is related to its mass m_0 at the rest relative to R_j^n by:

$$m\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right] = \frac{m_0}{\sqrt{1-\left(\dfrac{v_1^{O_j}(t_j;t_{j0})}{c_{ij}}\right)^2}}. \tag{31}$$

The equations (30) and (31) have been derived by starting with the time explanation in the Newtonian sense.

6. FORCE AND CO-ORDINATE TRANSFORMATION

The general case

Theorem 13. Let two identical particles 1 and 2 have the same mass m_0 at the rest relative to a reference integral space TxR^n. Let another two integral spaces T_ixR_i^n and T_jxR_j^n be accepted, $T_i \neq T_j$. Let the space co-ordinate system R_i^n be the "centre-of-mass" frame for the particles 1 and 2. Let the space co-ordinate system R_j^n be at the rest relative to R^n. Then, a force $\mathbf{F}(.)$ acting on the particle 1, its speed $\mathbf{v}_1^{O_j}(t_j;t_{j0}) = v_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ and its acceleration $\mathbf{a}_1^{O_j}(t_j;t_{j0}) = a_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ relative to R_j^n are interrelated by:

$$\mathbf{F}\left[\mathbf{v}_1^{O_j}(t_j;t_{j0}), \mathbf{a}_1^{O_j}(t_j;t_{j0})\right] = \frac{m_0}{\left[1-\left(\dfrac{v_1^{O_j}(t_j;t_{j0})}{c^j}\right)^2\right]^{3/2}}\mathbf{a}_1^{O_j}(t_j;t_{j0}). \tag{32}$$

The special case

Theorem 14. Let two identical particles 1 and 2 have the same mass m_0 at the rest relative to a reference integral space TxR^n. Let another two integral spaces

$T_i\text{x}R_i^n$ and $T_j\text{x}R_j^n$ be accepted, $T_i = T_j$. Let the space co-ordinate system R_i^n be the "centre-of-mass" frame for the particles 1 and 2. Let the space co-ordinate system R_j^n be at the rest relative to R^n. Then, a force **F**(.) acting on the particle 1, its speed $\mathbf{v}_1^{O_j}(t_j;t_{j0})$ = $v_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ and its acceleration $\mathbf{a}_1^{O_j}(t_j;t_{j0}) = a_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ relative to R_j^n are interrelated by:

$$\mathbf{F}\left[\mathbf{v}_1^{O_j}(t_j;t_{j0}), \mathbf{a}_1^{O_j}(t_j;t_{j0})\right] = \frac{m_0}{\left[1-\left(\frac{v_1^{O_j}(t_j;t_{j0})}{c_{ij}}\right)^2\right]^{3/2}}\mathbf{a}_1^{O_j}(t_j;t_{j0}). \quad (33)$$

The equations (32) and (33) have been derived by starting with the time explanation in the Newtonian sense.

7. ENERGY AND CO-ORDINATE TRANSFORMATION

The expressions for the energy E of a body now result from its general expression $E=mc^2$ and the expression for the corresponding mass depending on the body velocity $\mathbf{v}_1^{O_j}(t_j;t_{j0})$ relative to the reference integral space $T_j\text{x}R_j^n$ with respect to which the body mass at the rest is m_0.

The general case

Theorem 15. Let two identical particles 1 and 2 have the same mass m_0 at the rest relative to a reference integral space $T\text{x}R^n$. Let another two integral spaces $T_i\text{x}R_i^n$ and $T_j\text{x}R_j^n$ be accepted, $T_i \neq T_j$. Let the space co-ordinate system R_i^n be the "centre-of-mass" frame for the particles 1 and 2. Let the space co-ordinate system R_j^n be at the rest relative to R^n. Then, the energy $E\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right]$ of the particle 1 moving with the speed $\mathbf{v}_1^{O_j}(t_j;t_{j0})$ = $v_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ is related to its mass m_0 and to its energy $E_0 = m_0(c^j)^2$ when it is at the rest with respect to R_j^n by:

$$E\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right] = \frac{m_0(c^j)^2}{\left[1-\left(\frac{v_1^{O_j}(t_j;t_{j0})}{c^j}\right)^2\right]^{3/2}} = \frac{E_0}{\left[1-\left(\frac{v_1^{O_j}(t_j;t_{j0})}{c^j}\right)^2\right]^{3/2}}. \quad (34)$$

The special case

Theorem 16. Let two identical particles 1 and 2 have the same mass m_0 at the rest relative to a reference integral space $T\text{x}R^n$. Let another two integral spaces $T_i\text{x}R_i^n$ and $T_j\text{x}R_j^n$ be accepted, $T_i = T_j$. Let the space co-ordinate system R_i^n be the "centre-of-mass" frame for the particles 1 and 2. Let the space co-ordinate system R_j^n be at the rest relative to R^n. Then, the energy $E\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right]$ of the particle 1 moving with the speed $\mathbf{v}_1^{O_j}(t_j;t_{j0})$ = $v_1^{O_j}(t_j;t_{j0})\mathbf{r}_0$ is related to its mass m_0 and to its energy $E_0 = m_0(c^j)^2$ when it is at the rest with respect to R_j^n by:

$$E\left[\mathbf{v}_1^{O_j}(t_j;t_{j0})\right] = \frac{m_0 c^{j^2}}{\left[1-\left(\frac{v_1^{O_j}(t_j;t_{j0})}{c_{ij}}\right)^2\right]^{3/2}} = \frac{E_0}{\left[1-\left(\frac{v_1^{O_j}(t_j;t_{j0})}{c_{ij}}\right)^2\right]^{3/2}}. \quad (35)$$

The equations (34) and (35) have been derived by starting with the time explanation in Newton's sense.

A material – energy explanation and interpretation of the preceding results can be searched along lines of [20].

8. PHYSICAL CONTINUITY AND UNIQUENESS PRINCIPLE (PCUP)

A scalar form of the Physical Continuity and Uniqueness Principle (PCUP for short) was introduced in [10]. It will be presented in the sequel in a vector form.

Physical Continuity and Uniqueness Principle. Vector form:

a) ***Physical Continuity Principle***

A vector variable, all the entries of which are physical variables, can change its vector value from one vector value to another one only by passing elementwise through all intermediate vector values.

b) ***Physical Uniqueness Principle***

A vector variable, all the entries of which are physical variables, possesses a unique local instantaneous real vector value at any place (in any being or in any object) at any moment.

The continuity of time is crucial for applications of the Physical Continuity and Uniqueness Principle. It is explained by analysing mathematical and physical responses of various non-linear functions in [10].

8. TIME, PCUP AND CONTROL OF MECHANICAL SYSTEMS

The following mathematical model is typical for mechanical systems (aeroplanes, robots, vehicles):

$$A[q(t)]q^{(2)}(t) + h[q(t), q^{(1)}(t)] = B[q(t)]u(t;.) + Dd(t),$$
$$y(t) = g[q(t)], \tag{36}$$

where

$$\det A(q) \neq 0, \forall q \in R^n, \tag{37}$$
$$q \in R^n, h(.): R^n \text{ x } R^n \text{ x } R^d \rightarrow R^n, d(.): R \rightarrow R^d,$$
$$B(.): R^n \rightarrow R^{n\text{x}r}, u(.): R \text{x} \ldots \rightarrow R^r, D \in R^{n\text{x}n}, \tag{38}$$
$$y \in R^m, g(.): R^n \rightarrow R^m, g(q) \in C^{(2)}(R^m).$$

Assumption 1.

a) All the system variables obey the Physical Continuity and Uniqueness Principle.
b) Every $d(.) \in S_d$ obeys the Physical Continuity and Uniqueness Principle.
c) Every $y_d(.) \in S_y$ and its first two derivatives obey the Physical Continuity and Uniqueness Principle.
d) The state vector $x(t) = [q^T(t) \;\; q^{(1)T}(t)]^T$ is elementwise measurable.

We demand that every system real output response $y(.;q_0, q_0^{(1)}; d, u)$, $y(t;q_0, q_0^{(1)}; d, u) \equiv y(t)$, tracks elementwise with a requested tracking quality any system desired output response $y_d(.) \in S_y$ for every perturbation vector function $d(.) \in S_d$. The requested tracking quality can be defined by properties of solutions to the following vector equation:

$$T[e^{(i)}(t), \ldots, e(t), \int_0^t e(t)dt] = 0 \text{ for all } t \in [s, \infty[,$$
$$\forall (e_0, e_0^{(1)}, \ldots, e_0^{(i-1)}) \in R^m \text{ x } R^m \text{ x} \ldots \text{x } R^m, \tag{39}$$
$$T(.): R^n \text{ x} \ldots \text{x } R^n \rightarrow R^n, \; i \in \{1, 2, \ldots\}$$

where e(.) is the output error vector function,
$e(t) = y_d(t) - y(t)$, $e(t) \in R^m$,
and $s \in R^+$ can be chosen arbitrarily small.

Real forms of the vector functions h(.) and d(.) are unknown as well as their real instantaneous vector values. Their variations are unpredictable. However, the matrix function B(.) and the vector function g(.) are completely known.

Let $N(.): R \rightarrow R^{m\text{x}m}$, $R_+ = [0, \infty[$,

$$N(t) = \text{diag}\{\eta_1(t) \;\; \eta_2(t) \;\; \ldots \;\; \eta_m(t)\} \in C^{(1)}(R_+), \tag{40}$$

$$\eta_i(t) \begin{cases} = 1, \; t = 0, \\ \text{is strictly monotonously decreasing}, \; t \in [0, s[, \\ = 0, \; t \in [s, \infty[, \end{cases} \tag{41}$$
$$\forall i = 1, 2, \ldots, m.$$

Let

$$p\left[e^{(i)}(t), \ldots, e(t), \int_0^t e(t)dt\right] = T\left[e^{(i)}(t), \ldots, e(t), \int_0^t e(t)dt\right] - N(t)T([e^{(i)}(0), \ldots, e(0), 0]). \tag{42}$$

A control is *natural tracking control* for system (36) - (38) on $S_d \text{x} S_y$ if and only if it is continuous in time, guarantees that the system exhibits the requested tracking for any its desired output response $y_d(.) \in S_y$ and for every $d(.) \in S_d$, and that it can be synthesised and implemented without using any information about real forms and vector values of h(.) and d(.).

Let $u(t^-) \equiv u(t - \varepsilon)$ for ε sufficiently small: $0<\varepsilon<<<1$ or for $\varepsilon \rightarrow 0^+$. In the ideal case $\varepsilon = 0^+$.

Theorem 17. Let Assumption 1 hold. In order for a control u(.) to be, in the ideal case or in the case ε is sufficiently small, a natural tracking control for system (36) - (38) on $S_d x S_y$ ensuring a requested tracking quality defined by (39) it is both necessary and sufficient that (43) and (44) hold:

$$\mathrm{rank}[J(q)A^{-1}(q)B(q)] = m, \forall q \in R^n, \tag{43}$$

$$u[t, e(t), q(t)] = u[t^-, e(t^-), q(t^-)] + G[q(t)]p\left[e^{(i)}(t), \ldots, e(t), \int_0^t e(t)dt\right], \tag{44a}$$

$$G(q) = [J(q)A^{-1}(q)B(q)]^T \bullet \left\{[J(q)A^{-1}(q)B(q)][J(q)A^{-1}(q)B(q)]^T\right\}^{-1}. \tag{44b}$$

9. CONCLUSION

It has been reproved that ***Newton's*** explanation of properties of *time* is correct. *Time* is a physical variable the value of which (called: moment or instant) has been strictly monotonously increasing independently of all other variables, processes, movements and events. ***Newton*** explained the absolute sense and meaning of time. However, ***Newton*** *himself* explained also a relative sense, meaning and use of time by referring to the use of different time units and scales in order to measure the time values and their differences (time intervals). A sequence of *time* values is used to determine the order of happenings of events, processes and/or movements. A time interval is used to measure a duration of a process, of a movement, of a rest or of the existence of a physical object. Time evolution expresses the ageing process. Speeds of evolution of physical processes can be different not only for different processes, objects and/or systems, but also for different parts of the same object and/or system. Their different evolutionary speeds can imply different time scales and units.

The Newtonian understanding of *time* has been used at first to generalise the Lorentz transformation. It has been exploited to establish new results on velocity, acceleration, mass and energy of a moving body, and on a force causing its movement. The new relationships are more general than those existing in the Lorentz – Einstein relativity theory. The former incorporates the latter. It has been also shown that by using the real features of *time* as explained essentially by ***Newton*** and in more details herein, we are able to deduce all the basic relationships of the relativity theory and to show that they correspond to a special, singular case.

A joint use of *time* and of a new physical principle: Physical Continuity and Uniqueness Principle, which has been introduced recently [10] and generalised to the vector form in the paper, is a basis for the synthesis of a natural tracking control of mechanical systems.

Two cases should be distinguished with regard to the relationship between the scaling factors in the co-ordinate transformations and a choice of an arbitrary point A and its position. If the former depend on the latter then it is the strict non-uniform transformation. Otherwise, it is non-uniform (or, even uniform) transformation. They open new directions in the relativity theory: *the strict non-uniform relativity theory* and *non-uniform relativity theory*. The obtained results represent their fundaments. Besides, they pose a question on *uniform relativity theory* in which time scales will be uniform over the space.

Bibliographical references

[1] R. D'Inverno, *Introducing Einstein's Relativity*, Clarendon Press, 0xford (1995).

[2] A. Einstein, *La théorie de la relativité*, Gauthier-Villars et Cie, Paris (1921).

[3] A. Einstein, *L'éther et la théorie de la relativité.* Gauthier-Villars et Cie, Paris (1921).

[4] A. Einstein, *The meaning of relativity.* Methuen & Co. Ltd., London (1950).

[5] A. Einstein, *Relativity.* University paperbacks, Methuen & Co. Ltd., London (1960).

[6] A. Einstein and L. Infeld, *L'évolution des idées en physique*, Flammarion, Paris (1938).

[7] Lj. T. Grujić, *Time and Modeling*, Proc. 1996 IEEE International Conference on Systems, Man and Cybernetics, Beijing, **3** (1996), 2438-2443.

[8] Lj. T. Grujić, *Multiple Time Scale Systems, Time and Modeling*, Pre-prints of the IFAC-IMACS-IFIP Conference on Control of Industrial Systems, Belfort, France, **1**, (1997), 293-298; also in Proceedings of the IFAC Conference Control of Industrial Systems, Pergamon, Elsevier, **1** (1997), 189-194.

[9] Lj. T. Grujić, A. A. Martynyuk and M. Ribbens-Pavella, *Large Scale Systems Stability under Structural and Singular Perturbations*, Springer Verlag, Berlin (1987).

[10] Ly. T. Gruyitch, *Physical continuity and uniqueness principle. Exponential natural tracking control,* Journal: Neural, Parallel and Scientific Computing, Dynamic Publishers, Atlanta, USA, Vol. 6, No. 2, (1998), 143 - 170.

[11] Ly. T. Gruyitch, "Time, Relativity and Physical Principle: Generalisations and Applications", *Proc. VI International Conference Physical*

Interpretations of Relativity Theory, London, (1998), 134-170.

[12] Ly. T. Gruyitch, "Time, Relativity and Physical Principle: Generalizations and Applications", Nelinijni Koluvanna, Kiev, Ukraine, **2**, No. 4, (1999), 465-489.

[13] R. E. Kalman, P. L. Falb and M. A. Arbib, *Topics in Mathematical System Theory*. Mc Graw Hill, New York (1969).

[14] H. A. Lorentz, *Versuch einer theorie der electricschen und optischen Erscheinungen in bewegten Körpern*, E. K. Brill, Leiden, (1895).

[15] H. A. Lorentz, *Das Relativitätsprinzip*, B. G. Teubner, Leipzig (1914).

[16] H. A. Lorentz, *The theory of Electrons*, B. G. Teubner, Leipzig (1916).

[17] H. A. Lorentz, *Deux mémoires de Henri Poincaré sur la physique mathématique*, Acta mathematica, Zeitscrift, **38** (1921), 294-308.

[18] H. A. Lorentz, A. Einstein and H. Minkowski, *Das Relativitätsprinzip*, B. G. Teubner, Leipzig (1923).

[19] H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, *The Principle of Relativity*, Dover Publications, Inc., New York (1952).

[20] P. Marmet, *Einstein's Theory of Relativity versus Classical Mechanics*, Newton Physics Books, 2401 Ogilvie Road, Gloucester, ON, Canada, K1J 7N4, 1997.

[21] E. F. Mishchenko, Yu. S. Kolesov, A. Yu. Kolesov, N. Kh. Rozov, *Asymptotic Methods in Singularly Perturbed Systems*, Consultants Bureau, New York (1994).

[22] D. S. Naidu, *Singular Perturbation Methodology in Control Systems*, Peter Peregrinus Ltd., London (1988).

[23] I. Newton, *Mathematical Principles of Natural Philosophy - BOOK I. The Motion of Bodies*, (The first edition 1687), William Benton, Publisher, Encyclopaedia Britannica, Inc., Chicago (1952).

[24] R. E. O'Malley, Jr., *Introduction to Singular Perturbations*, Academic Press, New York (1974).

[25] I. Prigogine, *Etude Thermodynamique des Phénomènes Irréversibles*, Dunod, Paris (1947).

[26] I. Prigogine, *Non - Equilibrium Statistical Mechanics*, Interscience Publishers, John Wiley & Sons, New York (1962).

[27] D. R. Smith, *Singular-perturbation theory*, Cambridge University Press, Cambridge (1985).

[28] A. N. Tikhonov, *Systems of differential equations containing small parameters in the derivatives*, Math. Sbor. (in Russian), **31**, (1952), 576 – 586.