PIRT Conference, Imperial Coll., London, Sept. 2000; Proceedings,(Late papers), ISBN 1 873 694 05 9, PD Publications (Liverpool), pp.16-25.

RELATIVITY BY CONFORMAL TRANSFORMATION AND HYPERBOLIC GEOMETRY

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Abstract

The writer's previous papers at these conferences [1],[2],[3],[4] have described aspects of the Varićak theory of Special Relativity in hyperbolic space [5],[6]. The present paper continues this subject.

Firstly, following introductory comments on the Lorentz group, we show that the Einstein composition law has wider validity than its normal statement implies. Secondly hyperbolic velocity is defined and it is shown how Newton's Law of Motion takes a simple and natural form when expressed in terms of rate of change of hyperbolic velocity. Thirdly we further develop the idea how the hyperbolic theory arises from the space-time metric by application of the ideas of projective geometry [7],[3]. From it is deduced the triangle of hyperbolic velocities expressing the Einstein composition law in the form giving relative hyperbolic velocities. This projective point of view gains significance by its relation to the axiomatic view of relativity due to Carathéodory [8], [9], [10] which leads to the assumption of a conformal condition under coordinate transformation. Using this conformal condition the paper shows that hyperbolic velocity is an absolute invariant under coordinate transformation making formulation of physical laws in terms of this quantity especially meaningful.

1. Lorentz Transformations

The investigations of Lorentz, Poincaré and Einstein [11], [12], [13], [14] dealt with transformations of the form:

$$x' = \kappa\ \gamma(x - vt)$$
$$y' = \kappa\ y$$
$$z' = \kappa\ z$$
$$t' = \kappa\ \gamma(t - (v/c^2)x)$$

where

$$\gamma = 1/\sqrt{(1-v^2/c^2)}$$

i.e. including an arbitrary nonzero constant κ which should be taken positive if time and space reversals are to be avoided. Poincaré originally defined the *Lorentz group* as the group generated by such transformations observing that this group includes:

a) Scalar multiplication.
b) Equations of the above form along the x, y and z axes with κ=1, (i.e. the pure Lorentz transformations),
c) Proper space rotations (i.e. conserving right handed axes)

This group might nowadays be called the homogeneous restricted Lorentz group augmented by the inclusion of scalar multiplications. It is these transformations which are of interest in this paper.

Any such transformation satisfies the equation

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \kappa^2(x^2 + y^2 + z^2 - c^2t^2)$$

Which implies that the sphere

$$x^2 + y^2 + z^2 = c^2t^2$$

describing the outward expansion of a light wave starting from the origin at time t zero, is transformed into a similar sphere by any transformation of the group. Conversely it may be shown that any linear transformation having this property must be of the above form.

As regards the constant κ, both Einstein and Poincaré considered that it should be taken equal to unity and to do so has since become standard practice. The reason for so doing is the reciprocity of the two frames of reference (for details see Pauli pp. 9-11 in the English version). Poincaré remarked that any transformation with κ not equal to unity can always be regarded as a transformation with κ put equal to unity followed by a scalar multiplication by κ. Nevertheless, for the present paper it is convenient to keep this constant in the equation.

2. Remarks on Einstein's velocity composition formula

Einstein's 1905 derivation of the composition formula introduced inertial frames here called K, K' with K' moving relative to K along the x-axis with uniform velocity v. The coordinates and times of the two frames are related by the above equations with κ =1. A uniform motion relative to frame K' in the plane z=0 represented by

$$x' = u_x' \ t' \qquad y' = u_y' \ t'$$

is transformed into a uniform motion in K of

$$x = \frac{u_x' + v}{\{1 + v\ u_x'/c^2\}} t$$

$$y = \frac{u_y'}{\gamma \ \{1 + v\ u_x'/c^2\}} t$$

from which the velocity components u_x, u_y in the K frame are identified as

$$u_x = \frac{u_x' + v}{\{1 + v\ u_x'/c^2\}}.$$

$$u_y = \frac{u_y'}{\gamma \ \{1 + v\ u_x'/c^2\}}.$$

From this follows the *Einstein composition formula*

$$u = \frac{\sqrt{\{v^2 + u'^2 + 2\ v\ u'\ \cos A - (v\ u'/c.\ \sin A)^2\}}}{1 + (v\ u'/c^2)\ \cos A}$$

where

$$u = \sqrt{(u_x^2 + u_y^2\}} \qquad u' = \sqrt{(u_x'^2 + u_y'^2)}$$

denote the magnitudes of the velocities and A the angle between them. For rectilinear motion along the x-axis, this formula becomes simply

$$u = \frac{v + u'}{1 + vu'/c^2} .$$

Concerning this derivation we comment that it is unnecessary initially to assume either that $\kappa = 1$ or that there is uniform motion. Because from the inverse of the initial equations in differential form:

$$dt = \kappa^{-1}\gamma \ (dt' + (v/c^2)\, dx')$$
$$dx = \kappa^{-1}\gamma \ (dx' + v\, dt')$$
$$dy = \kappa^{-1} \quad dy'$$

Division by dt immediately results in the above formulae relating components of u and u':

$$dx/dt = (dx'/dt' + v)/\{1 + (v/c^2)\ dx'/dt'\}$$
$$dy/dt = (dy'/dt')/\gamma\{1 + (v/c^2)\ dx'/dt'\}$$

Consequently Einstein's composition formula holds for any case when the initial differential form of the Lorentz equations holds. As the assumption is often made that these equations hold also for non-uniform motion, the validity of the composition formula for certain non-uniform motions looks likely. Einstein himself had of course used the Lorentz equations in his treatment of the accelerated motion of an electron.

As is well known, the concept of the sum of two velocities is ambiguous in Special Relativity and there are advantages in considering as basic the formula for the difference of two velocities, which is the inverse formula

$$u' = \frac{\sqrt{(v^2 + u^2 - 2\ vu \cos A - (vu \sin A /c)^2)}}{1 - vu/c^2 . \cos A}$$

The difference of the two velocities (here v, u) both taken relative to the same origin has the unambiguous meaning of a relative velocity. It has a clear geometrical meaning (see below).

3. Hyperbolic velocity and hyperbolic acceleration

The *rapidity* w is defined by

$$w = th^{-1}(v/c)$$

the principal value of the inverse hyperbolic function being taken which maps the range of values $-c < v < c$ on to the range of values $-\infty < w < \infty$. As is clear from the equation

$$th\ (w+w') = \frac{th\ w + th\ w'}{1 + th\ w.\ th\ w'} .$$

the composition of collinear rapidity's w and w' gives a rapidity $w + w'$.
Instead of rapidity w which is nondimensional it is more natural in dynamics to use the corresponding dimensional quantity:

$$V = c\ th^{-1}(v/c) = v + 1/3.(v/c)^2 + ...$$

This will be termed *hyperbolic velocity*. Following Varićak this is to be regarded as the true velocity from which the usual velocity v is found as a Euclidean projection. Hyperbolic velocity has the following properties:

(a) Unlike normal velocity, it can take any value from $-\infty$ to $+\infty$. and when $v \to \pm c$ then $V \to \pm \infty$, i.e. the hyperbolic velocity of light is infinite.

(b) When $v \ll c$, $V \approx v$ so then V becomes identified with normal velocity.

(c) Hyperbolic velocities combine by simple addition and this property extends to vector addition (see below).

The *hyperbolic acceleration* α is defined as rate of change of hyperbolic velocity with respect to proper time:

$$\alpha = dV/d\tau$$

The proper time τ here is also time observed relatively to the moving body. An important example of hyperbolic accelerated motion is given by

$$x = A \,\mathrm{ch}\, a\tau/c$$
$$ct = A \,\mathrm{sh}\, a\tau/c$$

A, a are constants. From this we find

$$dx = A \,\mathrm{sh}\, a\tau/c.\ d\tau/c$$
$$cdt = A \,\mathrm{ch}\, a\tau/c.\ d\tau/c$$

giving

$$dx/dt = c \,\mathrm{th}\, (a\tau/c)$$

so that the hyperbolic velocity is

$$V = a\tau$$

and the hyperbolic acceleration is

$$dV/d\tau = a = constant$$

4. Newton's Law of motion

Planck (1906, 1907) showed that in Special Relativity the expression for momentum in rectilinear motion is

$$p = \frac{mv}{\sqrt{(1-v^2/c^2)}} = m\,\frac{dx}{d\tau}$$

m denoting rest mass, and that Newton's law of rectilinear motion may be written in terms of it as

$$F = \frac{dp}{dt}$$

Substituting th V/c for v/c follows

$$p = \frac{mv}{\sqrt{(1-(v/c)^2)}} = mc\ sh\ V/c = mV\ \{1 + 1/6.(V/c)^2 + ...\}$$

From this we get

$$dp/dt = m\ ch\ V/c.\ dV/dt = m\ /\sqrt{\ (1-(v/c)^2\}}\ dV/dt = m\ dV/d\tau$$

Newton's law for rectilinear motion then takes the form

$$F = m\ \alpha$$

i.e.

force = rest mass x hyperbolic acceleration

If F is constant, hyperbolic acceleration α is constant with the value F/m so from above we see that if the hyperbolic velocity is initially zero then

$$x = x_0\ ch\ F\tau/mc$$
$$ct = x_0\ sh\ F\tau/mc$$

This is the *hyperbolic motion* of Born (1909)

Note that for rectilinear motion, the expression for energy takes the following simple form in terms of hyperbolic velocity:

$$E/c = \frac{mc}{\sqrt{(1-(v/c)^2)}} = mc\ ch\ V/c$$

5. Relative motion and the triangle of hyperbolic velocities

Einstein's composition law in the form given above for difference of velocities represented by v_1 and v_2 is

$$v = \frac{\sqrt{(v_1^2 + v_2^2 - 2\ v_1\ v_2\ cos\ A - (v_1\ v_2/c.\ sin\ A)^2)}}{1 - v_1/c.v_2/c.cos\ A}$$

From this we find

$$(1-v^2/c^2) = \frac{(1-v_1^2/c^2)\ (1-v_2^2/c^2)}{(1 - v_1/c.\ v_2/c.cos\ A)^2}$$

Inverting and taking square roots we get

$$\frac{1}{\sqrt{(1-v^2/c^2)}} = \frac{1}{\sqrt{(1-v_1^2/c^2)}}\frac{1}{\sqrt{(1-v_2^2/c^2)}} - \frac{v_1/c}{\sqrt{(1-v_1^2/c^2)}}.\frac{v_2/c}{\sqrt{(1-v_2^2/c^2)}}\ cos\ A$$

On substituting for hyperbolic velocity follows

$$ch\ V/c = ch\ V_1/c\ ch\ V_2/c - sh\ V_1/c\ sh\ V_2/c\ cos\ A$$

which is the 'cosine rule' in hyperbolic space of radius of curvature c giving the length V of one side of a triangle in terms of the lengths V_1, V_2 of the other sides and the included angle A. If the other sides are considered as vectors in hyperbolic space, the form here gives the third side as the magnitude of the *relative hyperbolic* *velocity* of C with respect to B which may be denoted by $\mathbf{V}_{C/B}$. The geometrical interpretation makes it obvious that this relative velocity is independent of the origin and in this sense is absolute. It is also clear that relative hyperbolic velocities can be added in the correct order.

The triangle of hyperbolic velocities is fundamental for the hyperbolic interpretation of Special Relativity. Varićak (1910) had originally stated it in terms of rapidities and this was also done previously by the writer. But it looks physically more meaningful when stated in terms of hyperbolic velocities when the radius of curvature of the hyperbolic space is c. In this case it is appropriate to quote Borel, who rediscovered this result in 1913 and concluded:

> *'The principle of relativity corresponds to the hypothesis that the kinematic space is a space of constant negative curvature, the space of Lobachevski and Bolyai. The value of the radius of curvature is the speed of light.'*

As has been observed above, the formula for composition of velocities does not necessarily rely on the assumption of uniform motions and this will be true also for the hyperbolic form here considered. This being so, it is interesting that from it can be derived other results, such as Einstein's aberration formula, which are normally considered to rest on the standard assumptions of the Special Theory.

6. Differential Minkowski space

From any point in Minkowski space, the set of differential four vectors (*dx, dy, dz, dt*) itself forms a Minkowski space. Such four vectors may be classified as *time-like* or *space-like* according to whether they lie or do not lie inside the differential Monge cone

$$c^2\, dt^2 = dx^2 + dy^2 + dz^2$$

which represents those differential four vectors having the speed of light. Differential four vectors for physically feasible motions will satisfy

$$(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 < c^2$$

i.e.

$$dx^2 + dy^2 + dz^2 < c^2\, dt^2$$

and so will be time-like. In applications it is the *forward light-cone* which is most of interest characterized by the last condition with dt>0.

The four vectors considered as vectors can be characterized by a magnitude represented by the proper time interval

$$d\tau = \sqrt{dt^2 - (dx^2+dy^2+dz^2)/c^2}$$

and direction given by the ratios

$$dx : dy : dz : dt$$

Such ratios for those differential vectors representing physical motions and so lying within the Monge cone have the projective geometry of a hyperbolic space with the equation of the Monge cone as absolute. In this space the Cayley-Klein metric w defining projective distance between two differential four vectors (dx_1, dy_1, dz_1, dt_1),(dx_2, dy_2, dz_2, dt_2) may be defined by

$$ch\ w = \frac{(c^2dt_1dt_2-dx_1dx_2-dy_1dy_2-dz_1dz_2)}{\sqrt{(c^2dt_1^2 - dx_1^2 - dy_1^2 - dz_1^2)}\sqrt{(c^2dt_2^2 - dx_2^2 - dy_2^2 - dz_2^2)}}.$$

$$= \frac{c^2dt_1dt_2 - d\mathbf{r}_1.d\mathbf{r}_2}{\sqrt{(c^2dt_1^2 - d\mathbf{r}_1.d\mathbf{r}_1)}\sqrt{(c^2dt_2^2 - d\mathbf{r}_2.d\mathbf{r}_2)}}.$$

$$= \frac{c^2 - \mathbf{v_1}.\mathbf{v_2}}{\sqrt{(c^2 - \mathbf{v_1}.\mathbf{v_1})}\sqrt{(c^2 - \mathbf{v_2}.\mathbf{v_2})}}.$$

This is the distance between $\mathbf{v_1}$ *and* $\mathbf{v_2}$ in the Beltrami-Klein model of three dimensional hyperbolic space using the spherical region

$$v = \sqrt{\mathbf{v}.\mathbf{v}} < c$$

(see [22],[23],[3]) Writing the last formula in the form

$$ch\ w = \frac{1 - v_1/c.\ v_2/c.\cos A}{\sqrt{(1 - v_1^2/c^2)}\sqrt{(1 - v_2^2/c^2)}}.$$

where A is the angle between $\mathbf{v_1}$ and $\mathbf{v_2}$, we see from the last section that it is identical with the formula for the triangle of rapidities and expresses the same result as the Einstein composition formula.

7. The light-cone condition and conformal transformation

Following the line of thought of Einstein and Poincaré we may ask under what conditions an infinitesimal spherical wave

$$dx^2 + dy^2 + dz^2 = (c.dt)^2$$

remains unaltered by transformation of variables. Physically this can be thought of as replacing the requirement of invariance of the observed spherical wave at time t by the requirement of invariance for the infinitesimal Huyghens wavelets from which the sphere was generated.

The condition for invariance will be the existence of a relation

$$[dx', dy', dz', dt']^T = L\ [dx, dy, dz, dt]^T$$

between corresponding differentials where L is a Lorentz transformation in the sense of Poincaré given above. Then we will have a condition

$$dx'^2 + dy'^2 + dz'^2 - c^2\ dt'^2 = \kappa^2(x, y, z, t)(dx^2 + dy^2 + dz^2 - c^2\ dt^2)$$

which will be called the *light-cone condition*. This occurred in the papers of Bateman and Cunningham (1909-1910) on the invariance of Maxwell's equations under certain nonlinear transformations they called spherical wave transformations (see [17],[18],[19]). It was subsequently used by Carathéodory [8],[9] in an axiomatic approach to Special Relativity in which this condition was derived from what was called the *condition for normal light propagation*, i.e. the condition that the propagation of light is observed similarly in arbitrary coordinate systems.

If following Carathéodory we assume that the coordinates were related by differentiable transformation equations:

$$x' = X(x,y,z,t)$$
$$y' = Y(x,y,z,t)$$
$$z' = Z(x,y,z,t)$$
$$t' = T(x,y,z,t)$$

then we require the Jacobian matrix to be a Lorentz transformation L in the sense of Poincaré defined above.

Since the light-cone condition can be written as

$$dx'^2 + dy'^2 + dz'^2 + (ic\ dt')^2 = \kappa^2(x, y, z, t)(dx^2 + dy^2 + dz^2 + (ic\ dt)^2)$$

it can be related to conformal transformation in four dimensions, a fact which leads to the classification of such transformations [8]. One basic conformal transformation is the inversion:

$$x' = x/R^2 \quad y' = y/R^2 \quad z' = z/R^2 \quad t' = t/R^2$$
$$R^2 = (x^2+y^2+z^2-(ct)^2)$$

also called 'spherical wave transformation'.

EXAMPLE (Bateman 1910) This is an example of a transformation to a non-inertial system which nevertheless satisfies the condition of normal light propagation.

$$t' = t\ (1 + f\ x)$$
$$x' = x\ (1 + f\ x) + 1/2\ f(-x^2 + c^2t^2 - y^2 - z^2)$$
$$y' = y\ (1 + f\ x)$$
$$z' = z\ (1 + f\ x)$$

where f is small so that quantities of the second order in f may be ignored. The conformal condition is satisfied with

$$\kappa(x, y, z, t) = 1 + f\ x$$

On substituting for t from the first equation into the second we find, to the first order in f,

$$x' = x + f/2\ (x^2 + t'^2 - y^2 - z^2)$$

If x, y, z are fixed, the point with coordinate x' moves with constant acceleration f. The relation between t and t' agrees with that given by Einstein [20] in his treatment of an accelerated system.

8. Coordinate invariance of hyperbolic velocity

Let us now consider a transformation to new coordinates with differential transformation

$$[dx',dy',dz',dt']^T = L\ [dx,dy,dz,dt]^T$$

satisfying the light-cone condition which will be written

$$c^2dt'^2-dx'^2-dy'^2-dz'^2=\kappa^2(x, y, z, t)\ (c^2dt^2-dx^2-dy^2-dz^2)$$

since we are interested in time-like four vectors. The cone of time-like four vectors will be transformed into another cone of time-like four vectors since $\kappa^2>0$. Moreover the forward cone dt > 0 transforms to the forward cone dt' > 0 by the orthochronous property of the Lorentz transformation L.

We now show that the Cayley-Klein metric remains absolutely invariant under the linear transformation of the differentials. Take the linear combination of two differential four vectors:

$$[dx, dy, dz, dt] = \lambda(dx_1, dy_1, dz_1, dt_1) + \mu\ (dx_2, dy_2, dz_2, dt_2)$$

which by linearity transforms to

$$[dx', dy', dz', dt'] = \lambda(dx_1', dy_1', dz_1', dt_1') + \mu\ (dx_2', dy_2', dz_2', dt_2')$$

By taking both λ and μ positive we can ensure that the linear combination lies on the segment joining the two points and hence is time-like from the convexity of the cone of time-like four vectors.

Let us now apply the light-cone condition. On equating coefficients of the terms in λ^2, $\lambda\mu$ and μ^2 we get

$$c^2dt_1'^2-dx_1'^2-dy_1'^2-dz_1'^2 = \kappa^2\ (c^2dt_1^2-dx_1^2-dy_1^2-dz_1^2)$$

$$c^2dt_1'dt_2'-dx_1'dx_2'-dy_1'dy_2'-dz_1'dz_2' = \kappa^2(c^2dt_1dt_2-dx_1dx_2-dy_1dy_2-dz_1dz_2)$$

$$c^2dt_2'^2-dx_2'^2-dy_2'^2-dz_2'^2 = \kappa^2(c^2dt_2^2-dx_2^2-dy_2^2-dz_2^2)$$

from which the invariance of the ratio defining the Cayley-Klein metric which defines rapidity follows.

The hyperbolic spaces in the original and transformed light-cones are correspondingly mapped on to one another isometrically. Thus these two hyperbolic spaces are isometric.

We have here shown that rapidity and hyperbolic velocity are independent of the coordinate system provided that the coordinate transformation lies in the class considered.

9. References

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