<u>The Ancient Philosophy of *Vis* and the Modern Science of Relativity</u>

S.R. Madhu Rao

334, TK Extension, [Behind B.Ed.College], Mysore - 570 009, INDIA.,

Current Email ID (preferred) : <*srmrao1@sancharnet.in* >

Alternative Email ID **:** < *smrp20@ sancharnet.in* >

==

Since times immemorial, people have believed that to every moving object avails a certain strength of motion determined by its mass and speed. This strength has variously been called "forcefulness", "vigor", "vitality", "energy", "***vis***" and so forth. However, since a body's speed is not absolute, but always necessarily referenced to some observing agency with its own state of motion, the notion of such a ***vis*** seems to be in rank dissonance with relativistic invariance at first sight. Yet reconciliation **--** nay, even mutual *support* **--** is possible between ***vis*** and relativity **!** To pull off this surprise, one needs to stop worrying over what is happening to one or other ***single*** moving body at a given instant of time. Instead, if the *vis*-related fortunes of all the bodies in a collection isolated from the rest of the universe are reckoned *together* **--** and if the general mathematical expression of a body's *vis* is suitably chosen **--** then any ***change*** occurring in the *cumulative vis of **all** the bodies in the collection* does come out with one and the same value for each and every observer calculating it, no matter how widely the observers' individual states of motion differ from one another. The caveat "if the expression of *vis* is suitably chosen" is also remarkable in itself, for the reason that it serves to *uniquely* identify the exact mathematical form of the correct ***vis***. Our discussions in the sequel are aimed primarily towards lay audiences, and are therefore presented in a semipopular style.

==

1. Introduction : a few recapitulatory quotes

What is *energy*? Well, "It is the capacity of a mechanism or agency to do *work*."

What is *kinetic* energy? "It is the capacity of a body to do *work* by virtue of its state of *motion*."

What is *work*? "It is the product of a *force* with the *displacement* suffered by its point of application."

What is the mathematical expression for kinetic energy? "It is one-half of the product of a body's *mass* with the *square* of its speed."

What is *relativity*? "It is the conviction that the behavior of nature in a given context as studied by independent observers in some widely differing states of motion can yet be expressed in one and the same mathematical *form*."

2. A simple example and the quandary it entails

Our quotes represent the responses of bright student-groups receiving elementary physics education across the world today.

These are affirmations built around a few handy catchwords -- *energy*, *work*, *kinetic energy*, and so forth. Have we assimilated their true import yet?

Let us consider an example to fix our ideas. A lot of ***chemical*** energy is stored in the propellants of a rocket. When you ignite these substances in the rocket's motor, you are transforming their energy into your ***vehicle****'s* kinetic energy. With your rocket steadily gaining enormous speeds, the ***kinetic*** *energy* in question soars upward by leaps and bounds.

All this looks okay, to be sure, to an onlooker ***O*** who stays at rest beside the vehicle's launching pad. But there can be a veteran astronaut ***A*** coasting in outer space off the launching pad with a speed parallel to the rocket's, but at the same time also somewhat higher than the modest maximum you have designed your rocket to ever attain. Does ***A***, too, find the rocket vehicle's speed rising? ***Not at all*** **!** To this astronaut ***A***, the rocket you have launched is precisely a ***retro***rocket. Together with its launching pad, says ***A***, your craft (albeit with nose pointing forward) was already drifting *rearward* with a large velocity ***before*** it was fired. This ongoing motion was sharply ***decelerated***, according to him, by the rocket's burning propellants **:** your craft started rapidly ***losing*** speed -- and ***losing*** kinetic energy -- only after those substances were ignited **!**

Given this perplexing situation, how can ***A*** figure out a reasonable way towards properly auditing the stupendous reserves of energy being shed by the fierce fiery stuff in your craft's combustion chamber?

And, above all, what about *relativity*? Clearly, relativity would be in jeopardy unless the two observers ***A*** and ***O*** are able to present mutually reconcilable energy audits **!**

How they can indeed do so is a quandary most textbooks in the past have quietly preferred to skip by. But in doing so their authors have also thrown to the winds back a wonderful windfall opportunity -- the opportunity of bringing home the philosophy of *genuine* relativity at the grass-roots level of ordinary experience.

3. Relativity's reconciliation

The lesson that the preceding section's quandary holds for thinking mortals is that the variations occurring in the kinetic energy of a ***single*** body viewed in isolation are never properly auditable. True, an eventual vehicle emptied of propellants but designed to carry some useful payload is a vital ingredient of the rocket example, but this vehicle is not its ***only*** ingredient. Indeed a huge mass of *exhaust gases* would be found rushing out of your rocket's nozzles for quite a while at the start, and these gases are entitled to as much attention from observers as the simple vehicle alone is commonly perceived to be. The ultimate recipe for sorting out perplexities consists in mandatorily getting all our observers to keep track of the changes occurring in the kinetic energies of *both the vehicle* ***and*** *the gases*, and in the end to collate their findings in an honest and natural manner.

Suppose the burning propellants cause an increase $V_{\boldsymbol{O}}$ & $V_{\boldsymbol{A}}$ in the vehicle's kinetic energy, and an increase G_O & G_A in the kinetic energy of the gases -- $V_{\boldsymbol{O}}$ and G_O being the launch pad observer ***O***'s measurements, and likewise $V_{\boldsymbol{A}}$ and G_A being the veteran astronaut ***A***'s measurements[1]. While it is true that $V_{\boldsymbol{A}}$ differs vastly from $V_{\boldsymbol{O}}$ -- and that G_A likewise differs vastly from G_O -- the triumph of relativity consists in the remarkable fact that nature *always* reconciles the *sum* $V_{\boldsymbol{A}}+G_A$ with the *sum* $V_{\boldsymbol{O}}+G_{\boldsymbol{O}}$ **:** [2]

$$V_{\boldsymbol{O}}+G_{\boldsymbol{O}} = V_{\boldsymbol{A}}+G_A.$$

Thus, all observers obtain one and the same invariant value when they calculate the ***change*** occurring in the *total* kinetic energy of the composite dynamical *system* consisting of both your rocket vehicle and its exhaust gases.

Inasmuch as this fundamental invariance is not a widely publicized topic in basic physics courses, readers might well wish to be briefed on how it comes about mathematically from other known facts of elementary Newtonian mechanics. A short incidental Appendix at the end of this article addresses this issue -- in a dynamical setting much wider than might be apparent at first sight from a limited rocket scenario. The generality of its considerations should bring home the profound universal sweep of our invariance result.

4. A peep into history

The current essay began with a few quotes concerning *energy*, *work* and *relativity*. If you were to accept these pronouncements at their face value, you would quickly reach the conclusion that, as a category in physics, *work* should be more primitive than *energy* --

in fact more primitive than all energy including kinetic energy. However, the foundations of post-medieval scientific mechanics were laid by Galileo Galilei (1564-1642), Gottfried Wilhelm von Leibniz (!646-1716) and, above all, Isaac Newton (1642-1727), and none of the trio appears to have employed the notion of work in his writings in the sense of our relevant preliminary quote.

The term *work* is a later addition, made in 1826, to the physicist's vocabulary. To be more specific, it was Jean Victor Poncelot (1788-1867) who introduced it for the first time as a facile mathematical tool for computing the changes of energy occurring in various dynamical situations[3]. And, in this background, it is the definition of *work* as currently familiar that seems to have been fashioned from the expression for the *energy* of a body's motion, *already long since known and independently validated*, rather than the other way around.

As a matter of fact, the latter idea of energy -- or *vis* or *vitality* or *forcefulness* (as it has variously been called) -- dates back well into *antiquity*. It is true that the talk about *vis* started as a vague, murky and fuzzy idiom, and this idiom was to remain steeped in eerie mysticism for centuries on end. Perhaps it has to be so with all profound thoughts -- more or less. Even today a "completely precise" definition of energy seems to elude us, though, for sure, people *have* gained a fairly reliable feel for what its stuff consists in on the basis of a few good *working criteria*.

Viewed against this background, one shouldn't be surprised that a serious debate was raging in the sixteenth and seventeenth centuries as to which specific mathematical expression *correctly* represented the *vis* (or energy) of the state of motion of a given body in terms of the *mass* M and the *speed* S of that body. Before Leibniz came on the scene, René Descartes (1596-1650) with his followers had been inclined to the view[4] that the requisite correct expression was simply MS (= mass times speed [5]). Leibniz hotly contested this position, and projected what he called

$$\textbf{\textit{vis viva}} = MS^2$$

(= mass times speed *squared*) to be the genuine measure of the "vigor" involved in a body's motion. Still other expressions like MS^3, Me^S etc. were possible, of course, but fell through merely for want of articulate proponents.

5. Is the issue of "correct *vis* expression" decidable?

This story could evoke amused smiles from listeners in modern times, whose educational background has weaned them away to a lofty view that issues such as MS versus MS^2 can be merely "metaphysical", and not scientifically decidable one way or the other. "It all depends on the context in which the question is raised," one is told. "*Vis* or vitality being basically an imprecise term, MS^2 might appear to have better virtues than MS in some contexts, and it could be vice versa in others."

The fact, however, is that there could have been thinkers even in Leibniz's time who held this "modern" view. Maybe, Leibniz was at pains precisely to address these uncommitted ones when he marshaled his masterly defense of *vis viva*'s exclusive legitimacy. We cannot go into details here, but let us note that, far from being metaphysical, his arguments in this context were pegged[6] to an eminently respectable principle of high science -- the impossibility of *perpetuum mobile* under earth's gravity.

The impact of Leibniz's defense was certainly less spectacular than what he might have expected in his time. Nevertheless, thanks to the reconciliation brought about between Newtonian and Leibnizian systems of dynamics by later ***mathematicians***, Leibniz's *vis viva*, renamed for minor historical reasons as *kinetic energy*[7], does find a place of honor in the classical mechanics of our own times as well.

The secure accommodation of what essentially is Leibniz's *vis viva* in today's classical mechanics does not mark the happy ending of our story. Rather, this is its happy *beginning* **!** After all, the current writing is concerned with *relativity*, and no theme in this venture can reach its proper completion before unveiling the sparkling glow of *relativity* exuding from it.

6. The robotic slingshot in outer space

The crucial fact now is that, rather than appealing to the impossibility of *perpetuum mobile* under the action of earth's gravity, Leibniz could have found a vastly more telling defense of his *vis viva* in the universal principle of *relativity* instead **!**

For a simple illustration of this big surprise, imagine a robotic slingshot floating all by itself in outer space alongside an orbiting satellite in which Descartes and Leibniz are riding together as fellow-astronauts. . A little stone attached to the slingshot's well-stretched exotic rubber band is released by remote control from the spacecraft. Descartes and Leibniz now notice the stone and the rest of the slingshot flying apart in opposite directions[8].

Suppose the mass of the slingshot together with the stone securely held in the contraption was originally M (> 0). At the start, this M was at rest relative to the astronaut duo, and so would be entirely devoid of any *vis* -- any kinetic energy -- in their reckoning **:**

vis of slingshot and stone held together = 0, (*agrees* Descartes, *agrees* Leibniz). (D1 **and** L1)

Next suppose (as determined by both Descartes and Leibniz) the mass of the released stone is M_1 (> 0), its speed is S_1, the mass of the slingshot separated from the stone is M_2 (> 0), and M_2's speed is S_2. What then would be the **total** *vis* of the hurled stone and the rest of the slingshot? Naturally, the predilections of Descartes and Leibniz on the *vis* issue give different estimates of this quantity **:**

total *vis* of slingshot and stone flying apart = $M_1S_1 + M_2S_2$ (swears Descartes) **;** (D2)
total *vis* of slingshot and stone flying apart = $M_1S_1^2 + M_2S_2^2$ (swears Leibniz) **.** (L2)

Meanwhile, a certain Descartes-fan DF and a certain Leibniz-fan LF back on planet earth measure the speed of their revered mentors' spacecraft coasting overhead. Suppose they find this speed to be S. This self-same S would of course also be the speed, relative to DF and LF, of the slingshot-stone artifact flying alongside the spacecraft in the sky until the Descartes-Leibniz duo set it off with their remote-controlled signals. Accordingly[9],

vis of slingshot and stone held together = MS (swears DF) **;** (D3)
vis of slingshot and stone held together = MS^2 (swears LF) **.** (L3)

What happens after the signals actuate the slingshot? We have seen that the artifact then effectively breaks up into two separate pieces that fly in opposite directions with the speeds S_1 and S_2, but these speeds are as observed by Descartes and Leibniz from their spacecraft. At the moment, however, our interest has switched towards what speeds -- say, S'_1 and S'_2 -- their *fans* down on earth are going to observe. Holding over the small-order latter-day corrections unveiled by Albert Einstein (1879-1955) for discussion in Sec.9, we may go along for the time being with just the good old commonsense view in this matter, that

$$S'_1 = S - S_1 \quad \text{and} \quad S'_2 = S + S_2. \qquad \text{(\underline{Galilean})}$$

We have assumed here, of course, that the stone is hurled from the slingshot in the direction exactly *opposite* to the spacecraft's motion as observed from the earth.

The fans DF and LF have by this time already got busy working out the *vires* of the stone and the rest of the slingshot flying one behind the other with the different -- but not opposite **!** -- speeds S'_1 and S'_2. Their findings are

total *vis* of slingshot and stone flying apart = $M_1S'_1 + M_2S'_2$ (swears DF) **;** (D4)
total *vis* of slingshot and stone flying apart = $M_1S'^2_1 + M_2S'^2_2$ (swears LF) , (L4)

where the observed -- "primed" -- speeds are the ones given by just the equation (Galilean) displayed above.

7. *Vis* as mass times speed : A failed prophesy

Be ready now to witness the inexorable impact of primordial relativity as it unfolds itself in nature's simplest transactions. It is pretty clear that the mechanism of the slingshot is able to create anew a certain amount of ***additional*** "*vis*" when it is actuated by remote signals. Precisely how much is this amount?

It is (D2) minus (D1) by Descartes' computation, (D4) minus (D3) by his fan DF's, (L2) minus (L1) by Leibniz's computation, and, lastly, (L4) minus (L3) by his fan LF's.

And Nature puts its foot down and issues a clear-cut decree in this context. It is that**:**

< *he whose reckoning fails to agree* ***with his own fan's*** *is a false prophet* **! >**

To put it more prosaically, whenever the ***total*** *vis* of an isolated mechanism of several interacting bodies changes, the amount of its *change* as calculated by outside observers in diverse states of motion must yet work out to a necessarily identical value -- else *whatever sacred tenet it is that guides their calculations must be in serious error* **!** One may call this philosophical diktat the grand invariance of total ***vis variations*** in dynamics.

Accordingly, our worthy Descartes can be a true prophet on the *vi*s dialectic only if

$$(D4) - (D3) = (D2) - (D1).$$

Readers taking the trouble of reducing this condition to a manageable form by making use of all our preceding equations [including (Galilean)] will come up with the simplification

$$(M_1 + M_2 - M)\, S = 2\, M_1\, S_1. \qquad (\textbf{?D?})$$

In this relationship, the parameters M, M_1, M_2 and S_1 are the results of specific measurements made on a specific slingshot from a specific spacecraft carrying Descartes and Leibniz, and are therefore just *constants*. On the other hand, the lonely S is the result of fan DF's and fan LF's speed measurement targeted on the Descartes-Leibniz spacecraft. Instead of being stationed on the earth, these fans could well have been on the moon, or on an asteroid, or on one or other of many artificial satellites circling the earth in diverse orbits. The speed S in Eq.(**?D?**), therefore, can be regarded as a *variable* admitting a host of different possible values. The only way (**?D?**) can still come out true in this particularly harsh circumstance is by imposing the twin stipulations

$$\text{(i)} \quad M_1 + M_2 - M = 0 \quad \text{and} \quad \text{(ii)} \quad M_1\, S_1 = 0. \qquad \text{(Descartes corollaries)}$$

Condition (i) looks eminently reasonable, but, since $M_1 > 0$, condition (ii) implies in turn $S_1 = 0$.

What does $S_1 = 0$ *mean*? Remember, S_1 was the speed attained by the stone, as Descartes and Leibniz noticed from their spacecraft, when it was hurled from the slingshot contraption by remote control. Surely the duo would have observed the released stone flying past their craft with *some* little speed at least. Thus, the foregoing implication $S_1 = 0$ flies in the face of the *reality* $S_1 > 0$ actually witnessed by Descartes and Leibniz.

This unforeseen -- and astonishing -- conflict with direct experience irretrievably compromises Descartes' claim to prophethood in dynamics. His mass times speed *can**not*** possibly serve as *vis* -- not under the regime of the grand invariance mentioned above **!**

8. *Vis* as mass times *speed-squared* : A resounding success

The regime of grand invariance proves capable of rejecting not only MS (mass times speed), but MS^3, MS^4, Me^S -- and a host of many, many other apparent possibilities besides -- as no more than fake representations of "*vis*". In fact, it can be shown[10] that this invariance principle eliminates *each and every* candidate

$$M\,F(S) \quad (= \text{ mass times a mathematical } \textit{function} \text{ of speed})$$

that comes with its F(S) ***not*** expressible in the specific form

$$F(S) = \alpha\, S^2 + \beta, \quad (\alpha \text{ and } \beta \text{ any two constants})$$

-- assuming that speeds in various directions are summed up like simple *vectors* in just the pre-Einsteinean commonsense tradition[11].

Yet the intriguing surprise is that the invariance principle *allows* the expression $M\,F(S)$ with $F(S) = \alpha\, S^2 + \beta$ as a valid representation of *vis*. Leibniz's own

$$vis\ viva = MS^2$$

is a canonical example (α=1, β=0), and accordingly its rôle as a genuine representation of M's *vis* is validated as well.

How is it that MS^2 succeeds where MS does not? Remember, in Sec.7, MS failed the invariance test (D4)–(D3) = (D2)–(D1). As can be checked up from Secs.6 and 7 with a little mathematical effort, the *corresponding* criterion

$$(L\,4) - (L\,3) = (L\,2) - (L\,1).$$

for MS^2 simplifies to

$$(M_1 + M_2 - M)\, S = 2\,(M_1\, S_1 - M_2\, S_2). \qquad \textbf{(?L?)}$$

With S as its ***sole*** *variable* (cf. remarks made in Sec.7), this Eq.(**?L?**) can come true if and only if

$$\text{(i)}\quad M_1 + M_2 = M \quad \text{and} \quad \text{(ii)}\quad M_1\, S_1 = M_2\, S_2. \qquad \text{(\underline{Leibniz corollaries})}$$

The experiences of the astronaut duo, Descartes and Leibniz aloft in their spacecraft, now completely corroborate not only (i) but even (ii) as well. Stipulation (i) says that the arithmetically added separate masses of the hurled stone and the rest of the slingshot must exactly equal the latter bodies' single mass as it had been found before when they were held together. Stipulation (ii) says that the mutually opposed speeds S_1 of the hurled stone and S_2 of the recoiling slingshot (as determined from the spacecraft) must be inversely proportional to their respective masses[12]. These are conditions so universally familiar at present that few people in contemporary times would ever grudge nodding in agreement **!**

Some time back, we asked how much additional *vis* emerged from the mechanism of the slingshot when it was set off. The answer, of course, is

$$(L\,2) - (L\,1) \quad [= (L\,4) - (L\,3)]$$

in the Leibniz scenario, and this is simply (L 2) because (L 1) is zero (see beginning of Sec.6). As such,

$$\underline{\text{additional Leibnizian } vis \text{ created by the slingshot mechanism}} = (L\,2) = \underline{M_1 S_1^2 + M_2 S_2^2}.$$

Since we have ensured that this quantity is ***invariant***, we can now at last let it stand alone all by itself. It no longer cries for a *qualification* as to ***who*** made the measurements needed for its computation.

9. Newer developments with Einstein on the scene

True, our foregoing discussion has sprung surprises, but isn't it likely that the breath-taking advances of science made since Leibniz's bygone era -- and especially the phenomenal twentieth century developments -- would after all have ***dated*** its simple theme much too far beyond redemption?

No. That this theme has been quietly -- and successfully -- warding off philosophical and scientific obsolescence is yet another surprise not noticed by many even today ! □

As we have seen, the regime of grand invariance in dynamics (see Sec.7) all by itself leads to Leibniz's *vis viva* as essentially the only valid mathematical representation possible of a moving body's "strength" of motion. But this is contingent on the acceptance of pre-Einsteinean commonsense view on how speeds in various directions are to be added together. To uphold *vis viva* (= mass times speed-squared), one perforce needs to compound these speeds like simple *vectors*.

Sec.6's

$$S'_1 = S - S_1 \text{ and } S'_2 = S + S_2. \qquad \text{(Galilean)}$$

afford a rather trivial example of this commonsense tradition of manipulating speeds, but these expressions are the farthest we can manage to talk about within the constraints of the current essay. The architect of special relativity, Albert Einstein, found in 1905 that the formulas (Galilean) -- and their more general versions as well -- are actually somewhat in error, and need to be corrected wherever very high speeds prevail. Einstein's corrections of (Galilean), for instance, are the following **:**

$$S'_1 = (S - S_1) / (1 - SS_1 / c^2) \text{ and } S'_2 = (S + S_2) / (1 + SS_2 / c^2), \qquad \text{(Einsteinean)}$$

c being the speed of light *in vacuo*.

Suppose we plod again through all our earlier calculations, scrupulously employing (Einsteinean) now wherever (Galilean) was used before. Then we find, alas, that even Leibniz's *vis viva* MS^2 *promptly* fails its invariance test (L 4)−(L 3) = (L 2)−(L 1) **!**

The impact of Einstein's corrections renders Leibniz as luckless as Descartes **!**

10. Special relativity's new *vis*

Take heart. The simple truth is that special relativity's speeds with their distinctive style of adding to or subtracting from one another cry for an altogether ***new*** *vis* tailored differently to suit their radical needs. The very existence of such a new *vis* -- a *new* kinetic energy formulation -- will at once hand in a powerful boost as much to the scientific status of relativity as to the philosophical legitimacy of the *vis* dialectic.

It is true that in the years past people have groped, more or less, towards the requisite relativistic *vis* by various murky routes[13]. But, if only the grand invariance of total *vis variations* (Sec.7) had received wider recognition, nothing in this connection could have been more straightforward than a direct mathematical *search* for the new *vis* steered by this principle itself **!** It can be demonstrated[14] that such a search strategy homes in *straight* on the familiar specific expression

$$M\,[\,\alpha\,\gamma(S) - \beta\,],\quad [\alpha \neq 0,\ \beta\ \text{constants}\ ;\ \gamma(S) = 1 / \sqrt{(1 - S^2/c^2)}\,] \qquad \text{(\underline{Einstein's } \underline{\textit{vis}})}$$

as the ***only*** representation possible in special relativity of a (so-called "rest"-)mass M's *vis* if it is to support the grand invariance principle of Sec.7. Isn't this truly a moment of triumph for both the ancient philosophy of *vis* and the modern science of relativity?

We have seen that the grand invariance of total *vis variations* extracts the implications

$$\text{(i)}\ \ M_1 + M_2 = M \ \text{ and }\ \text{(ii)}\ \ M_1 S_1 = 0. \qquad \text{(\underline{Descartes corollaries})}$$

from Descartes' *vis* MS (Sec.7), and

$$\text{(i)}\ \ M_1 + M_2 = M \ \text{ and }\ \text{(ii)}\ \ M_1 S_1 = M_2 S_2. \qquad \text{(\underline{Leibniz corollaries})}$$

from Leibniz's *vis* MS^2 (Sec.8). While (Leibniz corollaries) are found justified to a high degree of accuracy, the second Descartes corollary $M_1 S_1 = 0$ is downright *false*. The question now arises as to what implications (Einstein's *vis*) leads to under the regime of the very same grand invariance. On working out, these implications turn out to be

$$\text{(i)}\ M_1\gamma(S_1^2) + M_2\gamma(S_2^2) = M \ \text{ and }\ \text{(ii)}\ M_1\gamma(S_1^2)\,S_1 = M_2\gamma(S_2^2)\,S_2. \quad \text{(\underline{Einstein corollaries})}$$

Therefore, at the behest of grand invariance, Einstein warns you not to expect M_1 and M_2 to add up to M when M breaks up into M_1 and M_2 **!** Perhaps this is the biggest surprise of his relativity theory.

11. *Vis* versus "timeward-directed" momentum component

At the end of Sec.8, we calculated, in the Leibnizian scenario, the amount of (the invariant) additional *vis* that the slingshot mechanism creates when it is set off. This amount turned out to be $M_1S_1^2 + M_2S_2^2$.

What is the corresponding result in the Einsteinean scenario? Here again a little peculiarity surfaces that doesn't seem to have received much attention. Take a look at (Einstein's *vis*) cited above with its two constants α and β. On working out straight from this expression, and using as well the (Einstein corollary) (i), it now turns out -- as a culmination of some not inconsiderable mathematical effort **!** -- that

$$\underline{\text{additional Einsteinean } \textit{vis} \text{ created by the slingshot mechanism}} = \beta\,(M - M_1 - M_2)$$
$$= \beta\,[\,M_1\{\gamma(S_1^2) - 1\} + M_2\{\gamma(S_2^2) - 1\}\,].$$

The peculiarity lies in the unceremonious *exit* of the first constant α into oblivion **!** The disappearance is baffling, yet can't be helped. □ The last expression can be expandedin a (binomial) series

$$\tfrac{1}{2}\,(\beta/c^2)(\,M_1S_1^2 + M_2S_2^2 + \text{higher order terms}\,).$$

Comparing with the good old Leibnizian result $M_1S_1^2 + M_2S_2^2$ recalled above, we infer that β needs to be set equal to $2c^2$ in order that Einstein's *vis* and Leibniz's *vis* may be expressed in *matching units*. (By the same token, of course, $\beta = c^2$ if the latter-day "*kinetic energy*" takes the place of Leibniz's *vis viva*.)

As for α, it simply doesn't *matter* which value you give it -- as long as that value isn't zero. This conclusion looks puzzling, to be sure, in the face of extensive contemporary relativity literature seeming to predicate $\beta = 0$ and $\alpha = c^2$ instead. The explanation is that the self-same expression cited above as (Einstein's *vis*) comes with the potential to play dual rôles that are somewhat mutually incompatible. In one of these rôles, as just seen, it serves as a representation of *vis* -- or kinetic energy -- in an ingenuous and rudimentary sense, and, in this scenario, α ***is*** immaterial and β ***must be*** $2c^2$ (or simply c^2) for the sake of continuance of the old-time measuring units.

The very same expression $M\{\alpha\gamma(S) - \beta\}$'s ***other*** rôle is inspired not so much by pristine philosophy as by sophisticated mathematics. Here it is poised to appear as a *vector component* -- the time-related component of a certain "vector", known as *four-momentum*, securely lodged in what is called a four-dimensional space-time**!** An apt description of this vector *component* could have been "timeward-directed *momentum*", but in the literature it, too, has acquired just the handy nomenclature of *energy* -- or relativistic "total" energy to connote the inclusion of both its so-called kinetic and other forms. It then turns out that, for this "relativistic energy" $M\{\alpha\gamma(S) - \beta\}$ to properly fit its latter rôle as the time-related *component of a four-momentum*, it ***has*** to go with $\beta = 0$ -- as well as $\alpha = c^2$ to ensure continuance of historical units.

In summary, the values of α and β depend thus on the context in which one chooses to make use of the expression we have been referring to as (Einstein's *vis*). Needless to say, the current writing is concerned with (Einstein's *vis*) in its rôle as *vis* proper -- vitality or forcefulness of motion -- rather than as a momentum component. This makes our resolve to go along with $\beta = 2c^2$ (or $\beta = c^2$) and $\alpha \neq 0$ positively unwavering.

12. Concluding remarks

Under the long-prevalent influence of the Newtonian tradition, the preferred choice of authors to approach the rudiments of relativistic mechanics has almost exclusively been via the four-momentum idiom. True, this ingenious theoretical construct is in consonance with relativity by being invariant for all observers. In any dynamical situation, not only the *cumulative* four-momentum of *all* the relevant bodies, but in fact even the *individual* four-momentum of each *single* body turns out to be relativistically invariant.

Yet this four-momentum's *mathematics* is sophisticated, and first-time learners struggling to cope with it could well miss the wood for the trees. The current writing has pointed to an attractive alternative to approach relativity-driven mechanics. Somehow this new approach, with the invariance of the cumulative *kinetic energy* (or *vis*) ***variations*** in closed dynamical systems as its basis, has remained practically unexplored hitherto in the history of science. Freer accessibility to non-specialist audiences puts its philosophical superiority in clear evidence. A further virtue of this invariance is that every result of the traditional four-momentum heuristics (primarily the *conservation* of cumulative vectorial four-momentum) does derive from the new approach as well -- in a more natural and systematic way, in fact.

A psychological hangover from earlier mechanics that oftentimes weighs heavily on the momentum dialect is the idea of "forces" mysteriously coming into play between spatially separated bodies. With the collapse of absolute simultaneity in relativistic physics, predications of such "actions at a distance" are no longer tenable, and, our invariance principle brings their inconsistency with relativity into sharper focus by explicitly envisaging concurrent observations made by multiple observers in vastly different states of motion.

In the context of forbidding actions at a distance, the invariance approach broadly commends the world view that any complex interaction in nature must be built up as a succession of elementary interactions. Each elementary interaction may be visualized as a single *event* -- one that takes place ***at*** a single location in space and ***at*** a single instant of time. This single event may consist in a mass M present at the spatial location in question breaking up into several fragments M_1, M_2, **...** that fly away in different directions -- or, inversely, several masses M_1, M_2, **...** coming together at that site from different directions and fusing up into a single mass M there. This simple and guileless imagery precludes actions instantly leaping across voids in space.

We have reached the end of our brief survey of the *vis* philosophy of dynamics. This philosophy is not the sort to lose its way in a morass of confounding metaphysical verbiage. It comes endowed with a clear direction. It lends itself to precise mathematical analysis. It has inspired great thinkers in the past. Descartes and Leibniz are but two outstanding examples.

And its crowning glory is that, in a superb show of unity between an ancient idea and a modern discovery, *it even blends miraculously with Albert Einstein's* ***relativity*** **!**

----

13. Appendix : Grand invariance principle's classical justification

Imagine a collection of masses M_1, M_2, ... interacting with one another, but isolated from any influence of the rest of the universe. Let, as measured by some observing agency O, the (vectorial) velocities of these masses be $\mathbf{V}_1$, $\mathbf{V}_2$, ... Then the cumulative kinetic energy of all the masses turns out to be

$$K = \tfrac{1}{2} (M_1 \mathbf{V}_1^2 + M_2 \mathbf{V}_2^2 + \dots)$$

by O's reckoning.

Now if O itself is moving relatively to another agency $O^{!}$ with a constant velocity $\mathbf{U}$, then $O^{!}$ measures the velocities of M_1, M_2, ... to be $\mathbf{V}_1 + U$, $\mathbf{V}_2 + \mathbf{U}$, ... , and their cumulative kinetic energy to be

$$K^{!} = \tfrac{1}{2} \{M_1 (\mathbf{V}_1 + \mathbf{U})^2 + M_2 (\mathbf{V}_2 + \mathbf{U})^2 + \dots \} .$$

As interactions progress, some of the masses may merge together, some may break up, yet others could stay intact. At a later time, therefore, O and $O^{!}$ would typically find a *different* number of *possibly different* masses, say m_1, m_2, ... Suppose the velocities of m_1, m_2, ... are $\mathbf{v}_1$, $\mathbf{v}_2$, ... as measured by O, and $\mathbf{v}_1 + \mathbf{U}$, $\mathbf{v}_2 + \mathbf{U}$, ... as measured by $O^{!}$. Their cumulative kinetic energy must then be

$$k = \tfrac{1}{2} (m_1 \mathbf{v}_1^2 + m_2 \mathbf{v}_2^2 + \dots)$$

according to O, and

$$k^{!} = \tfrac{1}{2} \{m_1 (\mathbf{v}_1 + \mathbf{U})^2 + m_2 (\mathbf{v}_2 + \mathbf{U})^2 + \dots \}$$

according to $O^{!}$.

It is easy to verify

$$(k^{!} - K^{!}) - (k - K) = \tfrac{1}{2} \mathbf{U}^2 \{ (m_1 + m_2 + \dots) - (M_1 + M_2 + \dots) \} + \mathbf{U} \cdot \{ (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + \dots) - (M_1 \mathbf{V}_1 + M_2 \mathbf{V}_2 + \dots) \}.$$

The laws of classical *mass* conservation and Newtonian *momentum* conservation now certify that the values of the differences appearing within the curly braces here are both zero. Hence the grand invariance --

$$k^{!} - K^{!} = k - K :$$

Both O and $O^{!}$ obtain the self-same value for the *increment* accruing to the cumulative kinetic energy of all the interacting masses present in whichever isolated collection they are currently observing.

Simple as it is, the foregoing argument exudes utmost generality. It applies as much to rocket propulsion as to any other process occurring in nature. In the case of a rocket, the masses involved are the *atoms* and *molecules* making up its propellants -- as well as those present in the walls of its combustion chamber, with which the former keep all the time vigorously colliding.

----

[1] In case of a decrease instead of increase, the corresponding measurement goes, of course, with a negative sign **;** also, our term "gas" is supposed to subsume prior forms of the same substance existing *before* the propellants' ignition.

[2] Incidentally, these quantities $V_{\boldsymbol{O}} + G_{\boldsymbol{O}}$ and $V_{\boldsymbol{A}} + G_A$ won't be equal to the *entire* chemical energy released by the burnt propellants. $V_{\boldsymbol{O}} + G_{\boldsymbol{O}}$ or $V_{\boldsymbol{A}} + G_A$ will only be equal to the propellants' chemical energy ***less the part of it wasted as heat***.

[3] Max von Laue, "Inertia and Energy" in *Albert Einstein* **:** *Philosopher-Scientist*, edited by P.A.Schilpp (Harper Torchbooks, New York, 1959), p.511.

[4] Max von Laue, op.cit., p.510.

[5] Descartes reckons his speed S without the association of any *direction* with it, so his product MS is not to be confused with Newton's vectorial *momentum* ("directed *quantity of motion*"). The latter, of course, is mass times *velocity*, not mass times speed.

[6] Rev.Fr. P.Castabel, *Leibniz and Dynamics*, English translation by R.E.W.Maddison, (Hermann, Paris, 1973).

[7] The impression that our "modern" kinetic energy (= **½** MS^2) differs significantly from Leibniz's *vis viva* -- by being only one-half of the latter -- is needlessly misplaced. The coefficient **½** entails nothing more than a simple change of *measuring units*.

[8] There is actually no need to take for *granted* that these two objects would fly away in exactly opposite directions (as observed by the astronauts looking out from inside their satellite). That they indeed would have to do so can be *proved* -- and proved even independently of Newton's laws. See the work cited below under Reference 10 in this connection.

[9] Masses such as M, M_1 and M_2 are supposed to be *invariant* for all observers. This remark applies, in this article, even within the realm of Einstein's special relativity, where any mass mentioned will have to be understood necessarily as a "rest" mass always.

[10] For a rigorous proof, see S.R.Madhu Rao, "The kinematical roots of dynamics", *Am. J. Phys.* **68** (4), 329-333, (April 2000).

[11] What happens under the Einsteinean rule of adding velocities naturally holds even greater significance. See S.R.Madhu Rao, op.cit., as well as Secs. 9 and 10 of the current article.

[12] This, of course, is nothing but the law conservation of momentum (or, equivalently, Newton's third law of motion). Readers must be careful to observe, however, that under our grand invariance philosophy, Newton's truths are *straight* not taken for granted. Instead, they are *deduced* as necessary logical consequences of the invariance *itself*. Thus, what is supposed to be reigning at the most fundamental level is the invariance, not Newton's laws.

[13] A widely noticed example appears to be G.N.Lewis and R.C.Tolman, *Phil.Mag.* **18**, 510, (1909), and versions deriving from it as offshoots have been presented in many relativity textbooks. See, for instance, A.P.French, *Special Relativity*, (MIT Cambridge, 1968), pp.169-175.

[14] See the work cited under Reference 10 above for full details.
