GENERAL RELATIVITY AS A FIELD THEORY ON A FIXED BACKGROUND AND MASSIVE GRAVITY

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In general relativity (GR), as known, energy-momentum and angular momentum are not localized due to the equivalence principle. However, the modern astrophysical and cosmological researches examine energy-momentum characteristics where the component of gravitational perturbations (in particular, waves) is essential. This situation requires a modified presentation of GR where energy, its density, etc. are defined by a reasonable way, unlike the standard geometrical description of GR. Here, we give a review of works where a such approach is developed. Thus, GR, without changing the physical sense, is interpreted as a theory of fields (including gravitational one) in a fixed arbitrary curved or flat spacetime. This presentation frequently is called as the field formulation of GR. Conservation laws based on a symmetrical (metric) energy-momentum tensor for all the fields are presented by conserved currents expressed through divergences of superpotentials (antisymmetric tensor densities). This form permits to connect a necessity to consider local properties of perturbations in applications with the academic imagination on the quasilocal nature of the conserved quantities in GR. The gauge invariance properties are studied and the non-localization is described explicitly in exact mathematical terms. The M/string considerations point out to possible modification of GR, for example, by adding "massive terms" including masses of spin-2 and spin-0 gravitons. A such original modification on the basis of the field formulation of GR is given by Babak and Grishchuk, and we present it here. They have shown that all the local weak-field predictions of the massive theory are in agreement with experimental data. Otherwise, the exact non-linear equations of the new theory eliminate the black hole event horizons and replace a permanent power-law expansion of the homogeneous isotropic universe with an oscillator behaviour. One variant of the massive theory allows "an accelerated expansion" of the universe.

1 Introduction

A majority of cosmological and astrophysical problems are studied in the framework of the perturbed approach. Investigations in gravitational physics, particularly in general relativity (GR), frequently are also carried out under assumption that perturbations of various kinds propagate in a background given (fixed) spacetime, which is a solution of the Einstein equations [1] - [5]. Both flat and curved background spacetimes, such as exact cosmological and black hole solutions, are exploited. As a rule, the perturbed Einstein equations are rewritten as follows. The linear in metric perturbation terms are placed on the left hand side; whereas all nonlinear terms are transported to the right hand side, and together with a matter energy-momentum tensor are treated as a total (effective) energy-momentum tensor $t^{(tot)}_{\mu\nu}$. This picture, *without changing the physical sense of GR*, however with different interpretation, was developed in a form of a theory of a tensor field with self-interaction in a fixed background spacetime; $t^{(tot)}_{\mu\nu}$ is obtained by variation of an action with respect to a background metric. Frequently it is called as a *field theoretical formulation* [6] of GR, we will call it simply as the *field formulation*. The history of these studies was begun in 50-th of XX century. Deser [7] generalized previous works and suggested the more clear presentation in Minkowski spacetime. We [8] developed the field formulation of GR on arbitrary curved backgrounds with all the properties of a field theory in a fixed spacetime. A related bibliography of earlier works particularly can be found in [6] - [8]. The foundation of the field formulation of GR and some references can be also found in the review and discussion works [9]. A more full, as we know, the modern bibliography can be found in [10, 11].

In our presentation at the conference PIRT-VI [12] we have reviewed already the field formulation of GR on *Ricci-flat* and *flat* backgrounds. We have demonstrated [12] advantages of this interpretation in some applications. Thus, it was given a description of asymptotically flat spacetimes at spatial infinity [13]; an analysis of particle trajectories approaching at the black hole horizons [14]; the energy distributions for the black hole solutions [15]; the re-presentation of the closed Friedmann world as a gravitational field configuration in a Minkowski space [16].

At the present, we give a review of a *recent* development of the field approach to GR. In section 2 a construction of the field formulation for *arbitrary curved* backgrounds is given. Just this is necessary for cosmological and astrophysical applications. In the last two-three decades the quasi-local approach, when energy-momentum and angular momentum became to be associated with finite spacetime domains, is developed very intensively (see the recent useful review by Szabados [17]). In contrast with this, in cosmological researches, e.g., it is necessary to consider and examine local perturbations. Therefore, one needs to connect the local description with the non-local characteristics. In section 3, on the basis of the works [18] - [20] we present the conserved currents (constructed on the basis of local densities) expressed through the superptentials, integration of which just leads to suface integrals (quasi-local conserved quantities). In section 4 we give a generalization of the results of sections 2 and 3 for various definitions of metrical perturbations and resolve related ambiguities. The one of more desirable properties is that an energy-momentum complex of a theory has to be free of the second (highest) derivatives of the field variables. The energy-momentum tensor in [8] does not satisfy this requirement. Babak and Grishchuk recently improved this situation [21]. Developing the approach [8] they reformulated the field interpretation of GR satisfying the above property; in section 5 we outline their approach and results. The non-localization problem in GR is discussed in exact mathematical terms in the framework of the field approach in section 5 too. The original and non-trivial technique developed in [21] is naturaly generalized by Babak and Grishchuk in [22] for constructing a gravitational theory with massive gravitons; we present and discuss these interesting results in section 6.

Let us give more general notations used in the paper.

- Greek indeces mean 4-dimesional spacetime coordinates. Small Latin indeces from the middle of alphabet i, j, k, ..., as a rule, mean 3-dimensional space coordinates; large Latin indeces A, B, C, ... are used as generalized ones for an arbitrary set of tensor densities, like Q^A. Usually $x^0 = ct$, where c is speed of light; $\kappa = 8\pi G/c^2$ is the Einstein constant; $(\alpha\beta)$ and $[\alpha\beta]$ are symmetrization and antisymmetrization in α and β.

- The dynamic metric in the Einstein theory, as usual, is denoted by $g_{\mu\nu}$ ($g = \det g_{\mu\nu}$), whereas $\overline{g}_{\mu\nu}$ ($\overline{g} = \det \overline{g}_{\mu\nu}$) is the background metric; $\eta_{\mu\nu}$ is a Minkowskian metric. A hat means that a quantity "$\hat{Q}$" is a density of the weight +1, it can be $\hat{Q} = \sqrt{-g}Q$, or $\hat{Q} = \sqrt{-\overline{g}}Q$, or independently from metric's determinants, it will be clear from a context. A bar means that a quantity "$\overline{Q}$" is a background one. Particular derivatives are denoted by (∂_i) and (∂_α); (D_α) is a covariant derivative with respect to $g_{\mu\nu}$ with the Chistoffel symbols $\Gamma^\alpha_{\beta\gamma}$; $(\overline{D}_\alpha)$ is a background covariant derivative with respect to $\overline{g}_{\mu\nu}$ with the Chirstoffel symbols $\overline{\Gamma}^\alpha_{\beta\gamma}$; $\delta/\delta Q^A$ — the Lagrangian derivative; $\pounds_\xi Q^A = -\xi^\alpha \overline{D}_\alpha Q^A + Q^A|^\alpha_\beta\, \overline{D}_\alpha \xi^\beta$ — the Lie derivative of a generalized tensor density Q^A with respect to the vector ξ^α.

- $R^\alpha{}_{\mu\beta\nu}$, $R_{\mu\nu}$, $G_{\mu\nu}$, $T_{\mu\nu}$ and R are the Riemannian, Ricci, Einstein and matter energy-momentum tensors and the curvature scalar for the physical spacetime; $\overline{R}^\alpha{}_{\mu\beta\nu}$, $\overline{R}_{\mu\nu}$, $\overline{G}_{\mu\nu}$, $\overline{T}_{\mu\nu}$ and $\overline{R}$ are the Riemannian, Ricci, Einstein and matter energy-momentum tensors and the curvature scalar for the background spacetime.

2 Perturbed Einstein equations (the field formulation) on arbitrary curved backgrounds

There are different possibilities to approach at the field formulation of GR. In [12] already we have presented some of them in detail. Let us shortly list them. The Deser principle [7] can be formulated as follows. *(i) The source of the linear massless field of the spin 2 is to be the total symmetrical (metric) energy-momentum tensor of all the fields, including gravitational one.* Just on the basis of this principle it was obtained the generalizations in [8] and it was given the presentation in [12]. The other known method was considered by many of authors and more clearly was presented in the work by Grishchuk [6]. Shortly it can be formulated as *(ii) a logic transformation from gravistatic (Newton law) to gravidynamics (Einstein equations).* The next principle is based on gauge properties of the Einstein theory. Thus stating a non-standard *(iii) 'localization' of Killing vectors of the background spacetime* it was developed the field formulation of GR in the work [23]. The way that has a more evident connection with the ordinary geometrical formulation is based on simple *(iv) decompositions of the usual variables of GR into background variables and perturbations.* Below in this section, using the results of the work [24] we present it detaily.

The main point is in the construction of the Lagrangian for the perturbed system. Consider the Einstein theory with the usual action

$$S = \frac{1}{c}\int d^4x \hat{\mathcal{L}}^E \equiv -\frac{1}{2\kappa c}\int d^4x \hat{R}(g_{\mu\nu}) + \frac{1}{c}\int d^4x \hat{\mathcal{L}}^M(\Phi^A,\ g_{\mu\nu}) \tag{2.1}$$

where for the sake of simplicity one assumes that $\hat{\mathcal{L}}^M(\Phi^A,\ g_{\mu\nu})$ depends on the first derivatives only, and the Einstein equations together with the matter ones:

$$\frac{\delta \hat{\mathcal{L}}^E}{\delta \hat{g}^{\mu\nu}} = -\frac{1}{2\kappa}\frac{\delta \hat{R}}{\delta \hat{g}^{\mu\nu}} + \frac{\delta \hat{\mathcal{L}}^M}{\delta \hat{g}^{\mu\nu}} = 0\,, \tag{2.2}$$

$$\frac{\delta \hat{\mathcal{L}}^E}{\delta \Phi^A} = \frac{\delta \hat{\mathcal{L}}^M}{\delta \Phi^A} = 0\,. \tag{2.3}$$

Now, define the metric and matter perturbations as

$$\sqrt{-g}g^{\mu\nu} \equiv \hat{g}^{\mu\nu} \equiv \overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu}\,, \qquad \Phi^A \equiv \overline{\Phi^A} + \phi^A\,. \tag{2.4}$$

The background system is described by the Lagrangian:

$$\overline{S} = \frac{1}{c}\int d^4x \overline{\hat{\mathcal{L}}^E} \equiv -\frac{1}{2\kappa c}\int d^4x \overline{\hat{R}} + \frac{1}{c}\int d^4x \overline{\hat{\mathcal{L}}^M}\,, \tag{2.5}$$

and the background quantities $\overline{\hat{g}^{\mu\nu}}$ and $\overline{\Phi^A}$ satisfy the corresponding background equations:

$$-\frac{1}{2\kappa}\frac{\delta \overline{\hat{R}}}{\delta \overline{\hat{g}^{\mu\nu}}} + \frac{\delta \overline{\hat{\mathcal{L}}^M}}{\delta \overline{\hat{g}^{\mu\nu}}} = 0, \qquad \frac{\delta \overline{\hat{\mathcal{L}}^M}}{\delta \overline{\Phi^A}} = 0\,. \tag{2.6}$$

The perturbations $\hat{l}^{\mu\nu}$ and ϕ^A now are thought as independent dynamic variables. The perturbed system is to be described by a corresponding Lagrangian on the background of the system (2.5) and (2.6). After substituting the decompositions (2.4) into the Lagrangian of the action (2.1) a such Lagrangian (so-called *dynamical Lagrangian* [24]) is presented as follows

$$\hat{\mathcal{L}}^{dyn} = \hat{\mathcal{L}}^E(\overline{g} + l, \overline{\Phi} + \phi) - \hat{l}^{\mu\nu}\frac{\delta\overline{\hat{\mathcal{L}}^E}}{\delta\overline{\hat{g}^{\mu\nu}}} - \phi^A\frac{\delta\overline{\hat{\mathcal{L}}^E}}{\delta\overline{\Phi^A}} - \overline{\hat{\mathcal{L}}^E} - \frac{1}{2\kappa}\partial_\alpha\hat{k}^\alpha = -\frac{1}{2\kappa}\hat{\mathcal{L}}^g + \hat{\mathcal{L}}^m \tag{2.7}$$

where zero's and linear in $\hat{l}^{\mu\nu}$ and ϕ^A terms of the functional expansion are subtracted from the Einstein Lagrangian. Zero's term is the background Lagrangian, whereas the linear term is proportional to the l.h.s. of the background equations (2.6). However, one should not to use the background equations in $\hat{\mathcal{L}}^{dyn}$ before its variation because really $\hat{\mathcal{L}}^{dyn}$ is quadratic and more in the fields $\hat{l}^{\mu\nu}$ and ϕ^A.

If one chooses the vector density

$$\hat{k}^\alpha \equiv \hat{g}^{\alpha\nu}\Delta^\mu_{\mu\nu} - \hat{g}^{\mu\nu}\Delta^\alpha_{\mu\nu} \tag{2.8}$$

with the definition

$$\Delta^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \overline{\Gamma}^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\rho}\left(\overline{D}_\mu g_{\rho\nu} + \overline{D}_\nu g_{\rho\mu} - \overline{D}_\rho g_{\mu\nu}\right)\,, \tag{2.9}$$

where the decomposition (2.4) is used, then a pure gravitational part in the Lagrangian (2.7) is

$$\begin{aligned}\hat{\mathcal{L}}^g &= \hat{R}(\overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu}) - \hat{l}^{\mu\nu}\overline{R}_{\mu\nu} - \overline{\hat{g}^{\mu\nu}R_{\mu\nu}} + \partial_\mu\hat{k}^\mu \\ &= -(\Delta^\rho_{\mu\nu} - \Delta^\sigma_{\mu\sigma}\delta^\rho_\nu)\overline{D}_\rho\hat{l}^{\mu\nu} + (\overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu})\left(\Delta^\rho_{\mu\nu}\Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma}\Delta^\sigma_{\rho\nu}\right)\,.\end{aligned} \tag{2.10}$$

It depends on only the first derivatives of the gravitational variables $\hat{l}^{\mu\nu}$. In the case of a flat background the Lagrangian (2.10) transfers to the Rosen covariant Lagrangian [25]. The matter part of (2.7) is

$$\hat{\mathcal{L}}^m = \hat{\mathcal{L}}^M\left(\overline{g} + l, \overline{\Phi} + \phi\right) - \hat{l}^{\mu\nu}\frac{\delta\overline{\hat{\mathcal{L}}^M}}{\delta\overline{\hat{g}^{\mu\nu}}} - \phi^A\frac{\delta\overline{\hat{\mathcal{L}}^M}}{\delta\overline{\Phi^A}} - \overline{\hat{\mathcal{L}}^M}\,. \tag{2.11}$$

The variation of an action with the Lagrangian $\hat{\mathcal{L}}^{dyn}$ with respect to $\hat{l}^{\mu\nu}$ and some algebraic calculations give the field equations in the form:

$$\hat{G}^L_{\mu\nu} + \hat{\Phi}^L_{\mu\nu} = \kappa\left(\hat{t}^g_{\mu\nu} + \hat{t}^m_{\mu\nu}\right) \equiv \kappa\hat{t}^{(tot)}_{\mu\nu}\,, \tag{2.12}$$

where the l.h.s. linear in $\hat{l}^{\mu\nu}$ and ϕ^A consists of the pure gravitational and matter parts:

$$\hat{G}^L_{\mu\nu}(\hat{l}) \equiv \frac{\delta}{\delta\overline{g^{\mu\nu}}}\hat{l}^{\rho\sigma}\frac{\delta\overline{\hat{R}}}{\delta\overline{\hat{g}^{\rho\sigma}}} \equiv \tfrac{1}{2}\left(\overline{D}_\rho\overline{D}^\rho\hat{l}_{\mu\nu} + \overline{g}_{\mu\nu}\overline{D}_\rho\overline{D}_\sigma\hat{l}^{\rho\sigma} - \overline{D}_\rho\overline{D}_\nu\hat{l}_\mu{}^\rho - \overline{D}_\rho\overline{D}_\mu\hat{l}_\nu{}^\rho\right), \tag{2.13}$$

$$\hat{\Phi}^L_{\mu\nu}(\hat{l},\phi) \equiv -2\kappa\frac{\delta}{\delta\overline{g^{\mu\nu}}}\left(\hat{l}^{\rho\sigma}\frac{\delta\overline{\hat{\mathcal{L}}^M}}{\delta\overline{\hat{g}^{\rho\sigma}}} + \phi^A\frac{\delta\overline{\hat{\mathcal{L}}^M}}{\delta\overline{\Phi^A}}\right)\,. \tag{2.14}$$

The r.h.s. of Eq. (2.12) is the symmetrical energy-momentum tensor density

$$\hat{t}^{(tot)}_{\mu\nu} \equiv 2\frac{\delta\hat{\mathcal{L}}^{dyn}}{\delta\overline{g^{\mu\nu}}} \equiv 2\frac{\delta}{\delta\overline{g^{\mu\nu}}}\left(-\frac{1}{2\kappa}\hat{\mathcal{L}}^g + \hat{\mathcal{L}}^m\right) \equiv \hat{t}^g_{\mu\nu} + \hat{t}^m_{\mu\nu}. \tag{2.15}$$

In expansions, $\hat{t}^{(tot)}_{\mu\nu}$ is not less than quadratic in $\hat{l}^{\mu\nu}$ and ϕ^A, the same as the Lagrangian $\hat{\mathcal{L}}^{dyn}$. The explicit form of the gravitational part is

$$\hat{t}^g_{\mu\nu} = \frac{1}{\kappa}\left[\left(-\delta^\rho_\mu\delta^\sigma_\nu + \tfrac{1}{2}\overline{g}_{\mu\nu}\overline{g}^{\rho\sigma}\right)\left(\hat{\Delta}^\alpha_{\rho\sigma}\Delta^\beta_{\alpha\beta} - \hat{\Delta}^\alpha_{\rho\beta}\Delta^\beta_{\alpha\sigma}\right) + \overline{D}_\tau\hat{Q}^\tau_{\mu\nu}\right]\,, \tag{2.16}$$

where

$$
\begin{aligned}
2\hat{Q}^{\tau}_{\mu\nu} \equiv & -\bar{g}_{\mu\nu}\hat{l}^{\alpha\beta}\Delta^{\tau}_{\alpha\beta} + \hat{l}_{\mu\nu}\Delta^{\tau}_{\alpha\beta}\bar{g}^{\alpha\beta} - \hat{l}^{\tau}_{\mu}\Delta^{\alpha}_{\nu\alpha} - \hat{l}^{\tau}_{\nu}\Delta^{\alpha}_{\mu\alpha} + \hat{l}^{\beta\tau}\left(\Delta^{\alpha}_{\mu\beta}\bar{g}_{\alpha\nu} + \Delta^{\alpha}_{\nu\beta}\bar{g}_{\alpha\mu}\right) \\
& + \hat{l}^{\beta}_{\mu}\left(\Delta^{\tau}_{\nu\beta} - \Delta^{\alpha}_{\beta\rho}\bar{g}^{\rho\tau}\bar{g}_{\alpha\nu}\right) + \hat{l}^{\beta}_{\nu}\left(\Delta^{\tau}_{\mu\beta} - \Delta^{\alpha}_{\beta\rho}\bar{g}^{\rho\tau}\bar{g}_{\alpha\mu}\right) .
\end{aligned} \tag{2.17}
$$

The matter part is expressed through the usual matter energy-momentum tensor $T_{\mu\nu}$ of the Einstein theory as

$$
\begin{aligned}
\hat{t}^{m}_{\mu\nu} = & \sqrt{-g}\left[\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\right)\left(T_{\rho\sigma} - \tfrac{1}{2}g_{\rho\sigma}T_{\pi\lambda}g^{\pi\lambda}\right) - \overline{T}_{\mu\nu}\right] \\
& - 2\frac{\delta}{\delta\overline{g^{\mu\nu}}}\left(\hat{l}^{\rho\sigma}\overline{\frac{\delta\hat{\mathcal{L}}^{M}}{\delta\hat{g}^{\rho\sigma}}} + \phi^{A}\overline{\frac{\delta\hat{\mathcal{L}}^{M}}{\delta\Phi^{A}}}\right) .
\end{aligned} \tag{2.18}
$$

At the usual description of GR the definition of the energy-momentum tensor by $\delta\hat{\mathcal{L}}^{E}/\delta g^{\mu\nu}$ is senseless because it is vanishing on the Eq. (2.2), whereas $\hat{t}^{(tot)}_{\mu\nu}$ is not vanishing on the field equations (2.12). A formal reason is that in the Lagrangian (2.7) the linear terms are subtracted.

By the definitions (2.14) and (2.18), the field equations (2.12) can be rewritten in the form:

$$
\hat{G}^{L}_{\mu\nu} = \kappa\left(\hat{t}^{g}_{\mu\nu} + \delta\hat{t}^{M}_{\mu\nu}\right) = \kappa\hat{t}^{(eff)}_{\mu\nu} \tag{2.19}
$$

where $\delta\hat{t}^{M}_{\mu\nu} \equiv \hat{t}^{M}_{\mu\nu} - \overline{\hat{t}^{M}_{\mu\nu}}$ is equal to $\hat{t}^{m}_{\mu\nu}$ in Eq. (2.18) without the second line. Thus $\delta\hat{t}^{M}_{\mu\nu}$ can be thought as a perturbation of $\overline{\hat{t}^{M}_{\mu\nu}} = \overline{\hat{T}^{M}_{\mu\nu}}$ which is not quadratic in the dynamic fields and does not follow from the Lagrangian (2.7). But now $\hat{t}^{(eff)}_{\mu\nu}$ is the sourse of the linear gravitational field only (see Introduction).

Demonstrate the equivalence with the Einstein theory. Transfer $\hat{t}^{(tot)}_{\mu\nu}$ to the l.h.s. of Eq. (2.12) and use the definitions (2.13), (2.14) and (2.15) with (2.7):

$$
\begin{aligned}
& \hat{G}^{L}_{\mu\nu} + \hat{\Phi}^{L}_{\mu\nu} - \kappa\hat{t}^{(tot)}_{\mu\nu} \equiv -2\kappa\frac{\partial\overline{\hat{g}^{\rho\sigma}}}{\partial\overline{g^{\mu\nu}}}\frac{\delta}{\delta\hat{l}^{\rho\sigma}}\left[-\frac{1}{2\kappa}\hat{R}\left(\overline{\hat{g}^{\alpha\beta}} + \hat{l}^{\alpha\beta}\right) + \hat{\mathcal{L}}^{M}\left(\overline{\Phi^{A}} + \phi^{A};\ \overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu}\right)\right] \\
& + 2\kappa\frac{\delta}{\delta\overline{g^{\mu\nu}}}\left(-\frac{1}{2\kappa}\overline{\hat{R}} + \overline{\hat{\mathcal{L}}^{M}}\right) .
\end{aligned} \tag{2.20}
$$

Because the second line is proportional to the operator of the background equations in (2.6), then Eq. (2.12), as seen, is the Einstein equations (2.2), only in the form with using the decompositions (2.4).

3 Conservation laws in the field formulation of GR

At the beginning we discuss differential conservation laws on Ricci-flat (including flat) backgrounds. One has to take into account $\overline{\Phi^{A}} \equiv 0$, $\overline{\hat{\mathcal{L}}^{M}} \equiv 0$, $\hat{\Phi}^{L}_{\mu\nu} \equiv 0$ and use $\delta\overline{\hat{R}}/\delta\overline{\hat{g}^{\mu\nu}} = 0$ instead of the background equations (2.6). Then the Lagrangian (2.7) is simplified to

$$
\hat{\mathcal{L}}^{dyn} = -\frac{1}{2\kappa}\hat{\mathcal{L}}^{g} + \hat{\mathcal{L}}^{m} = -\frac{1}{2\kappa}\hat{\mathcal{L}}^{g} + \hat{\mathcal{L}}^{M}\left(\phi^{A};\ \overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu}\right), \tag{3.1}
$$

and the field equations (2.12) transform into the form of Eqs. (2.20)

$$
\hat{G}^{L}_{\mu\nu} = \kappa\left(\hat{t}^{g}_{\mu\nu} + \hat{t}^{m}_{\mu\nu}\right) \equiv \kappa\hat{t}^{(tot)}_{\mu\nu} . \tag{3.2}
$$

Thus, for Ricci-flat backgrounds $\hat{t}^{(tot)}_{\mu\nu} = \hat{t}^{(eff)}_{\mu\nu}$, and Eqs. (2.20) and (3.2) have the form annonced in Introduction. Because in Eq. (3.2) $\overline{D}_{\nu}\hat{G}^{L\nu}_{\mu} \equiv 0$, a divergence of Eq. (3.2) leads to

$$
\overline{D}_{\nu}\hat{t}^{(tot)\nu}_{\mu} = 0 . \tag{3.3}
$$

If a Ricci-flat background has a Killing vector λ^α it is easily to construct integral (non-local) conservation law. Firstly, one has to construct the conserved current

$$\hat{\mathcal{J}}^\nu(\lambda) = \hat{t}_\mu^{(tot)\nu}\lambda^\mu := \overline{D}_\nu \hat{\mathcal{J}}^\nu(\lambda) \equiv \partial_\nu \hat{\mathcal{J}}^\nu(\lambda) = 0\,. \tag{3.4}$$

Consider a background 4-dimensional volume V_4, the boundary of which consits of timelike "wall" S and two spacelike sections: $\Sigma_0 := t_0 = \text{const}$ and $\Sigma_1 := t_1 = \text{const}$. Because the conservation law (3.4) is presented by the scalar density it can be integrated through the 4-volume V_4: $\int_{V_4} \partial_\mu \hat{\mathcal{J}}^\mu(\lambda) d^4x = 0\,.$ The generalized Gauss theorem gives

$$\int_{\Sigma_1} \hat{\mathcal{J}}^0(\lambda) d^3x - \int_{\Sigma_0} \hat{\mathcal{J}}^0(\lambda) d^3x + \oint_S \hat{\mathcal{J}}^\mu(\lambda) dS_\mu = 0 \tag{3.5}$$

where dS_μ is the element of integration on S. If in Eq. (3.5) $\oint_S \hat{\mathcal{J}}^\mu(\lambda) dS_\mu = 0\,$, then the quantity

$$\mathcal{P}(\lambda) = \int_\Sigma \hat{J}^0(\lambda) d^3x \tag{3.6}$$

is conserved on Σ restricted by $\partial\Sigma$, intersection with S. In the inverse case, Eq. (3.5) describes changing the quantity (3.6), that is its flux through $\partial\Sigma$. It can be also assumed $\partial\Sigma \to \infty$.

The differential conservation laws (3.3) have also a place for backgrounds presented by Einstein spaces in Petrov's defenition [26]: $\overline{R}_{\mu\nu} = \Lambda \overline{g}_{\mu\nu}$ where Λ is a constant (see [8, 27, 28]). For arbitary curved backgrounds there are no conservation laws, like (3.3). Indeed, in the general case $\overline{D}_\nu \left(\hat{G}_\mu^{L\nu} + \hat{\Phi}_\mu^{L\nu}\right) \neq 0$ in (2.12), and $\overline{D}_\nu \hat{G}_\mu^{L\nu} \neq 0$ in (2.19). The reason is that the system (2.7) interacts with a complicated background geometry determined by the background matter fields $\overline{\Phi^A}$. Many of cosmological solutions are just not of Petrov's type.

Conservation laws for arbitrary curved backgrounds and arbitrary displacement vectors ξ^α were constructed in [20]. With using the technique of canonical Nœther procedure developed in [29] applied to the Lagrangian (2.7)) it was obtained the identity:

$$\frac{1}{\kappa}\hat{G}_\nu^{L\mu}\xi^\nu + \frac{1}{\kappa}\hat{l}^{\mu\lambda}\overline{R}_{\lambda\nu}\xi^\nu + \hat{\zeta}^\mu \equiv \overline{D}_\nu \hat{I}^{\mu\nu} \equiv \partial_\nu \hat{I}^{\mu\nu}\,. \tag{3.7}$$

The superpotential has the form:

$$\hat{I}^{\mu\nu} \equiv \frac{1}{\kappa}\hat{l}^{\rho[\mu}\overline{D}_\rho \xi^{\nu]} + \hat{\mathcal{P}}^{\mu\nu}{}_\lambda \xi^\lambda \equiv \frac{1}{\kappa}\left(\hat{l}^{\rho[\mu}\overline{D}_\rho \xi^{\nu]} + \xi^{[\mu}\overline{D}_\sigma \hat{l}^{\nu]\sigma} - \bar{D}^{[\mu}\hat{l}_\sigma^{\nu]}\xi^\sigma\right) \tag{3.8}$$

and, thus, $\partial_{\mu\nu}\hat{I}^{\mu\nu} \equiv 0$. It generalizes the Papapetrou superpotential [31]; indeed for the translations in Minkowski space $\xi^\lambda = \delta^\lambda_{(\rho)}$ in the Lorentzian coordinates one gets

$$\hat{I}^{\mu\nu}_{(\rho)} = \hat{\mathcal{P}}^{\mu\nu}{}_\rho = \frac{1}{2\kappa}\partial_\sigma\left(\delta^\mu_\rho \hat{l}^{\nu\sigma} - \delta^\nu_\rho \hat{l}^{\mu\sigma} - \overline{g}^{\sigma\mu}\hat{l}^\nu_\rho + \overline{g}^{\sigma\nu}\hat{l}^\mu_\rho\right)\,. \tag{3.9}$$

The same superpotential (3.8) was constructed by us in [19] by the other way, namely, by the Belinfante simmetrization of the canonical system in [29]. The last term on the l.h.s. of (3.7) is

$$2\kappa\hat{\zeta}^\mu \equiv 2\left(\bar{z}^{\rho\sigma}\overline{D}_\rho \hat{l}^\mu_\sigma - \hat{l}^{\rho\sigma}\overline{D}_\rho \bar{z}^\mu_\sigma\right) - \left(\bar{z}_{\rho\sigma}\overline{D}^\mu \hat{l}^{\rho\sigma} - \hat{l}^{\rho\sigma}\overline{D}^\mu \bar{z}_{\rho\sigma}\right) + \left(\hat{l}^{\mu\nu}\overline{D}_\nu \bar{z} - \bar{z}\overline{D}_\nu \hat{l}^{\mu\nu}\right) \tag{3.10}$$

where $2\bar{z}_{\rho\sigma} \equiv -\pounds_\xi \overline{g}_{\rho\sigma}$, and, thus, disappears on the Killing vectors of the background.

To write out a physical (weak) conservation laws from the identity (3.7) one has to use the field equations, which we substitute in the form of Eq. (2.19):

$$\hat{I}^\mu \equiv \mathcal{T}^\mu_\nu \xi^\nu + \hat{\zeta}^\mu = \overline{D}_\nu \hat{I}^{\mu\nu} = \partial_\nu \hat{I}^{\mu\nu}\,. \tag{3.11}$$

The generalaized total energy-momentum tensor density is

$$\hat{\mathcal{T}}^\mu_\nu \equiv \hat{t}^{g\mu}_\nu + \delta\hat{t}^{M\mu}_\nu + \frac{1}{\kappa}\hat{l}^{\mu\lambda}\overline{R}_{\lambda\nu} = \hat{t}^{(eff)}_{\mu\nu} + \frac{1}{\kappa}\hat{l}^{\mu\lambda}\overline{R}_{\lambda\nu} \tag{3.12}$$

where $\hat{t}^{(tot)\nu}_\mu$ is exchanging with $\hat{t}^{(eff)\nu}_\mu$, and the interaction with the background geometry term, $\hat{l}^{\mu\lambda}\overline{R}_{\lambda\nu}$, is adding. Thus, $\mathcal{T}^\mu_\nu$ plays the same role as $\hat{t}^{(tot)\nu}_\mu$ in Eq. (3.4) if Killing vectors exist. However, the current $\hat{I}^\mu$ has a more general applicability, than $\hat{\mathcal{J}}^\mu$ in (3.4): it is conserved, $\overline{D}_\mu\hat{I}^\mu = \partial_\mu\hat{I}^\mu = 0$, on arbitrary backgrounds and for arbitrary ξ^α. It is important for models with cosmological backgrounds where, besides, not only the Killing vectors are used fruitfully (see, e.g., [30]).

Due to antisymmetry of the superpotential (3.9) the conserved quantity, like (3.6), is expressed over a surface integral

$$\mathcal{P}(\xi) = \oint_{\partial\Sigma} \hat{I}^{0k}(\xi)ds_k \tag{3.13}$$

where ds_k is the element of integration on $\partial\Sigma$. It is important expression because it connects a quantity $\mathcal{P}(\xi)$ obtained by integration of local densities with a surface integral playing a role of a quasi-local quantity (see discussion in Introduction).

4 The field formulation of GR with different decompositions

In GR, components of each of metrical densities

$$g^a \in g^{\mu\nu},\ g_{\mu\nu},\ \sqrt{-g}g^{\mu\nu},\ \sqrt{-g}g_{\mu\nu},\ (-g)g^{\mu\nu},\ \ldots \tag{4.1}$$

could be chosen as independent dynamic variables. In the terms of generalized variables (4.1) the action of GR (2.1) is rewritten as

$$S = \frac{1}{c}\int d^4x\hat{\mathcal{L}}^{E(a)} \equiv -\frac{1}{2\kappa c}\int d^4x\hat{R}(g^a) + \frac{1}{c}\int d^4x\hat{\mathcal{L}}^M(\Phi^A,\ g^a)\,. \tag{4.2}$$

Variation with respect to g^a gives the gravitational equations in a corresponding form instead of (2.2).

The perturbations could also be defined for each of metric variables in Eq. (4.1):

$$\left.g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu},\ \ \hat{g}^{\mu\nu} = \overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu},\ \ g^{\mu\nu} = \overline{g}^{\mu\nu} + r^{\mu\nu},\ \ \ldots\right\} \ \ :=\ \ g^a = \overline{g}^a + h^a\,. \tag{4.3}$$

For the decomposition (4.3) following to the rules of constructing the Lagrangian (2.7) one gets

$$\begin{aligned} \hat{\mathcal{L}}^{dyn}_{(a)} &= -\frac{1}{2\kappa}\hat{R}\left(\overline{g^a} + h^a\right) + \hat{\mathcal{L}}^M\left(\overline{\Phi^A} + \phi^A;\ \overline{g^a} + h^a\right) \\ &- h^a\left(-\frac{1}{2\kappa}\frac{\delta\overline{\hat{R}}}{\delta\overline{g^a}} + \frac{\delta\overline{\hat{\mathcal{L}}^M}}{\delta\overline{g^a}}\right) - \phi^A\frac{\delta\overline{\hat{\mathcal{L}}^M}}{\delta\overline{\Phi^A}} - \left(-\frac{1}{2\kappa}\overline{\hat{R}} + \overline{\hat{\mathcal{L}}^M}\right) - \frac{1}{2\kappa}\partial_\nu\hat{k}^\nu\,. \end{aligned} \tag{4.4}$$

Its variation with respect to h^a and some re-calculations give the Einstein equations in the form (2.12):

$$\hat{G}^{L(a)}_{\mu\nu} + \hat{\Phi}^{L(a)}_{\mu\nu} = \kappa\hat{t}^{(tot\ a)}_{\mu\nu}\,. \tag{4.5}$$

The total symmetrical energy-momentum tensor density is defined as usual:

$$\hat{t}^{(tot\ a)}_{\mu\nu} \equiv 2\frac{\delta\hat{\mathcal{L}}^{dyn(a)}}{\delta\overline{g^{\mu\nu}}}\,. \tag{4.6}$$

In the l.h.s. of Eq. (4.5), due to the background equations, h^a are changed by independent variables

$$\hat{l}^{\mu\nu}_{(a)} \equiv h^a\frac{\partial\overline{\hat{g}^{\mu\nu}}}{\partial\overline{g^a}}, \tag{4.7}$$

and, thus, the same operators (2.13) and (2.14) are applied to $\hat{l}^{\mu\nu}_{(a)}$.

For some of different decompositions (4.3): $g^a_1 = \bar{g}^a_1 + h^a_1$ and $g^a_2 = \bar{g}^a_2 + h^a_2$ the variables (4.7) differ one from another in the second order in perturbations: $\hat{l}^{\mu\nu}_{(a2)} = \hat{l}^{\mu\nu}_{(a1)} + \hat{\beta}^{\mu\nu}_{(a)12}$. Because differences inter the linear expressions of equations (4.5) the energy-momentum tensor densities $\hat{t}^{(tot\ a1)}_{\mu\nu}$ and $\hat{t}^{(tot\ a2)}_{\mu\nu}$ get the same differences too. For the case of flat backgrounds this fact was noted by Boulware and Deser [32].

For the system (4.4) the identity

$$\frac{1}{\kappa}\hat{G}^{L(a)\mu}_{\nu}\xi^\nu + \frac{1}{\kappa}\hat{l}^{\mu\lambda}_{(a)}\overline{R}_{\lambda\nu}\xi^\nu + \hat{\zeta}^\mu_{(a)} \equiv \partial_\nu \hat{I}^{\mu\nu}_{(a)} \tag{4.8}$$

takes a place and is exactly the identity (3.7) with exchanging $\hat{l}^{\mu\nu}$ with $\hat{l}^{\mu\nu}_{(a)}$ only. Substituting Eq. (4.5) into the identity (4.8) we obtain the conservation law in the form:

$$\hat{I}^\mu_{(a)} = \left(\hat{t}^{g(a)\mu}_{\nu} + \delta\hat{t}^{M(a)\mu}_{\nu} + \kappa^{-1}\hat{l}^{\mu\lambda}_{(a)}\overline{R}_{\lambda\nu}\right)\xi^\nu + \hat{\zeta}^\mu_{(a)} = \hat{\mathcal{T}}^\mu_{(a)\nu}\xi^\nu + \hat{\zeta}^\mu_{(a)} = \partial_\nu \hat{I}^{\mu\nu}_{(a)} \tag{4.9}$$

analogous to (3.11) and (3.12). Thus a family of conservation laws (4.9) presents a corresponding family of superpotentials, e.g., in the form of the Abbott-Deser type [27]:

$$\hat{I}^{\mu\nu}_{(a)} = \frac{1}{\kappa}\left(\hat{l}^{\rho[\mu}_{(a)}\overline{D}_\rho\xi^{\nu]} + \xi^{[\mu}\overline{D}_\sigma\hat{l}^{\nu]\sigma}_{(a)} - \bar{D}^{[\mu}\hat{l}^{\nu]\sigma}_{(a)}\xi_\sigma\right) \tag{4.10}$$

with changing in (3.8) $\hat{l}^{\mu\nu}$ by $\hat{l}^{\mu\nu}_{(a)}$. Indeed, the known Abbott-Deser superpotential inter just this family with the decomposition $g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$ through the transformation (4.7) and for (anti-)de Sitter's backgrounds. Superpotentials of this family differs one from another due to the difference in perturbations. Otherwise, in the works [18, 19], with the generalized Belinfane's technique it was "symmetrized" the conserved quantities by Katz, Bičhák and Lynden-Bell [29]. This method does not depend on the choice of the variables from the set (4.1) and gives *uniquelly* [20] the superpotential (3.8). Thus, theoretically on the level of superpotentials the Boulware-Deser ambiguity [32] is resolved in favour of $\hat{l}^{\mu\nu}$.

5 The energy-momentum tensor with the first derivatives only

In this section we present the interesting results by Babak and Grishchuk [21] annonced in Introduction. From the point of view of the author, as it follows from the Introduction, in the framework of the field approach it is necessary to consider *both* flat and curved backgrounds. Unlike this, Babak and Grishchuk are staing on the position that for the development of the field approach it is enough to exploit the flat background only. This position has a natural basis, an opinion to which has to be paid. Indeed, first, nowadays really in all the experiments the Minkowski metric is used as a background one only; second, the field formulation of GR with a flat backgound, as a rule, permits to describe arbitrary curved and topologically non-trivial solutions of GR (see, e.g., [14] - [16]). Thus, in the present and next sections it is used the condition $\overline{R}_{\alpha\rho\beta\sigma} = 0$.

Here, our presentation is technically simplyer than that of Babak and Grishchuk [21], although, of course, is equivalent to their one. Thus we repeat their calculations on the basis of formulae (2.7) - (2.20) simplified to the case of Eqs. (3.1) - (3.3). Also, in the work [21] as independent variables it is used $l^{\mu\nu} = \hat{l}^{\mu\nu}/\sqrt{-\overline{g}}$, whereas we use $\hat{l}^{\mu\nu}$. Leaving the field equations to be equivalent, this leads to a difference in the direct definitions of the energy-momentum tensors. However, with taking into account the field equations this difference disappears and does not influence on results and conclusions.

With using the definition (2.4), present the expression (2.9) through the gravitational variables $\hat{l}^{\mu\nu}$:

$$\Delta^\lambda_{\mu\nu} \equiv \frac{1}{2\sqrt{-g}}\left[g_{\mu\rho}\overline{D}_\nu\hat{l}^{\lambda\rho} + g_{\nu\rho}\overline{D}_\mu\hat{l}^{\lambda\rho} - g_{\mu\alpha}g_{\nu\beta}g^{\lambda\rho}\overline{D}_\rho\hat{l}^{\alpha\beta}\right.$$

$$+\quad \tfrac{1}{2}\left(g_{\alpha\beta}\delta^{\lambda}_{\mu}\overline{D}_{\nu}\hat{l}^{\alpha\beta} + g_{\alpha\beta}\delta^{\lambda}_{\nu}\overline{D}_{\mu}\hat{l}^{\alpha\beta} - g_{\alpha\beta}g_{\mu\nu}g^{\lambda\rho}\overline{D}_{\rho}\hat{l}^{\alpha\beta}\right)\Big] \tag{5.1}$$

where $g_{\mu\nu}$, $\hat{g}^{\mu\nu}$ and $\sqrt{-g}$ are thought with dependence on the definition (2.4). Substitituting Eq. (5.1) into Eq. (2.16) with (2.17) one finds that $\hat{t}^{g}_{\mu\nu}$ depends on the second derivatives of $\hat{l}^{\mu\nu}$. After using the field equations (3.2) one gets

$$\hat{t}^{\mu\nu}_{g} = \hat{t}^{\mu\nu}_{(g-red)} + Q^{\alpha\beta\mu\nu}(\hat{t}^{m}_{\alpha\beta} - \tfrac{1}{2}\overline{g}_{\alpha\beta}\hat{t}^{m\rho}_{\rho}) + (2\sqrt{-\overline{g}})^{-1}\overline{D}_{\alpha\beta}(\hat{l}^{\alpha(\mu}\hat{l}^{\nu)\beta} - \hat{l}^{\mu\nu}\hat{l}^{\alpha\beta}) \tag{5.2}$$

where the reduced part with the first derivatives only is

$$4\kappa\sqrt{-\overline{g}}\hat{t}^{\mu\nu}_{(g-red)} = 2\overline{D}_{\rho}\hat{l}^{\mu\nu}\overline{D}_{\sigma}\hat{l}^{\rho\sigma} - 2\overline{D}_{\alpha}\hat{l}^{\mu\alpha}\overline{D}_{\beta}\hat{l}^{\nu\beta} + 2g_{\alpha\beta}g^{\rho\sigma}\overline{D}_{\rho}\hat{l}^{\mu\alpha}\overline{D}_{\sigma}\hat{l}^{\nu\beta} + g_{\alpha\rho}g^{\mu\nu}\overline{D}_{\sigma}\hat{l}^{\alpha\beta}\overline{D}_{\beta}\hat{l}^{\rho\sigma}$$
$$- \quad 4g_{\beta\rho}g^{\alpha(\mu}\overline{D}_{\sigma}\hat{l}^{\nu)\beta}\overline{D}_{\alpha}\hat{l}^{\rho\sigma} + \tfrac{1}{4}(2g^{\mu\delta}g^{\nu\omega} - g^{\mu\nu}g^{\omega\delta})(2g_{\rho\alpha}g_{\sigma\beta} - g_{\alpha\beta}g_{\rho\sigma})\overline{D}_{\delta}\hat{l}^{\rho\sigma}\overline{D}_{\omega}\hat{l}^{\alpha\beta} \tag{5.3}$$

and

$$(\sqrt{-\overline{g}})^{2}Q^{\alpha\beta\mu\nu} \equiv \hat{l}^{\alpha(\mu}\hat{\overline{g}}^{\nu)\beta} + \hat{l}^{\beta(\mu}\hat{\overline{g}}^{\nu)\alpha} + \hat{l}^{\alpha(\mu}\hat{l}^{\nu)\beta} - \tfrac{1}{2}\hat{\overline{g}}^{\mu\nu}\hat{l}^{\alpha\beta} - \tfrac{1}{2}\hat{l}^{\mu\nu}\left(\hat{\overline{g}}^{\alpha\beta} + \hat{l}^{\alpha\beta}\right). \tag{5.4}$$

Thus, even after the reducing, the second derivatives of $\hat{l}^{\mu\nu}$ inter the last term of Eq. (5.2). In [21] it was suggested the original way to exclude them from the energy-momentum tensor without changing the field equations. The Lagrangian (2.10) was modified as follows

$$\hat{\mathcal{L}}^{g}_{(mod)} = \hat{\mathcal{L}}^{g} + \hat{\Lambda}^{\alpha\beta\rho\sigma}\overline{R}_{\alpha\rho\beta\sigma}\,. \tag{5.5}$$

This is a typical way of incorporating constraints (because $\overline{R}_{\alpha\rho\beta\sigma} = 0$) by means of the undetermined Lagrange multipliers. The multipliers $\hat{\Lambda}^{\alpha\beta\rho\sigma}$ form a tensor which depends on $\overline{g}^{\mu\nu}$ and $\hat{l}^{\mu\nu}$ (without their derivatives) and satisfy $\hat{\Lambda}^{\alpha\beta\rho\sigma} = -\hat{\Lambda}^{\rho\beta\alpha\sigma} = -\hat{\Lambda}^{\alpha\sigma\rho\beta} = \hat{\Lambda}^{\beta\alpha\sigma\rho}$. Thus, the field equations (3.2) do not change. Then, in a correspondence with the modified Lagrangian (5.5), the modified energy-momentum tensor density is

$$\kappa\hat{t}^{g\,\mu\nu}_{(mod)} = \kappa\hat{t}^{g\,\mu\nu} - \overline{D}_{\alpha\beta}\left(\hat{\Lambda}^{\mu\nu\alpha\beta} + \hat{\Lambda}^{\nu\mu\alpha\beta}\right) \tag{5.6}$$

instead of (2.16). The originally undetermined multipliers $\hat{\Lambda}^{\mu\nu\alpha\beta}$ will now be determined. They can be chosen in such a way that the remaning second derivatives in (5.2) can now be removed. The unique possibility is $\hat{\Lambda}^{\mu\nu\alpha\beta} = (4\sqrt{-\overline{g}})^{-1}\left(\hat{l}^{\alpha\nu}\hat{l}^{\beta\mu} - \hat{l}^{\alpha\beta}\hat{l}^{\mu\nu}\right)$. Thus the equations (3.2) are not changed, but they have to be rewritten in the form

$$\begin{aligned}\hat{G}^{\mu\nu}_{L(mod)} &\equiv \hat{G}^{\mu\nu}_{L} - 2\overline{D}_{\alpha\beta}\hat{\Lambda}^{(\mu\nu)\alpha\beta} \equiv \tfrac{1}{2}(\sqrt{-\overline{g}})^{-1}\overline{D}_{\alpha\beta}\left[(\overline{\hat{g}^{\mu\nu}} + \hat{l}^{\mu\nu})(\overline{\hat{g}^{\alpha\beta}} + \hat{l}^{\alpha\beta}) - (\overline{\hat{g}^{\mu\alpha}} + \hat{l}^{\mu\alpha})(\overline{\hat{g}^{\nu\beta}} + \hat{l}^{\nu\beta})\right]\\ &= \kappa\left(\hat{t}^{\mu\nu}_{g\,(mod)} + \hat{t}^{\mu\nu}_{m}\right) \equiv \kappa\hat{t}^{\mu\nu}_{(mod-tot)}.\end{aligned} \tag{5.7}$$

The r.h.s. defined as a metric energy-momentum tensor for the system (5.5) is the sourse for the generalazed d'Alembert operator (general wave operator). Thus the l.h.s. is not more linear in $\hat{l}^{\mu\nu}$. Because on the flat background the divergence of the l.h.s. is identically equal to zero, then $\overline{D}_{\nu}\hat{t}^{\mu\nu}_{(mod-tot)} = 0$. In Eqs. (5.7), $\hat{t}^{\mu\nu}_{(mod-tot)}$ can be reduced by the equations of motion, then they are rewritten as

$$\hat{G}^{\mu\nu}_{L(mod)} = \kappa\left[\hat{t}^{\mu\nu}_{(g-red)} + Q^{\alpha\beta\mu\nu}(\hat{t}^{m}_{\alpha\beta} - \tfrac{1}{2}\overline{g}_{\alpha\beta}\hat{t}^{m\rho}_{\rho}) + \hat{t}^{m}_{\mu\nu}\right] \equiv \kappa\hat{t}^{\mu\nu}_{(mod-tot-red)} \tag{5.8}$$

Thus, indeed on the equations of motion the energy-momentum tensor density in (5.8) is only with the first derivatives of gravitational variables. Again $\overline{D}_{\nu}\hat{t}^{\mu\nu}_{(mod-tot-red)} = 0$. Let us show that Eq. (5.8) is equivalent to the usual Einstein equations. Multiplying it by $\sqrt{-\overline{g}}$, and using the identification (2.4), the definition (2.18) for the flat background and the definition (5.4), in the Lorentzian coordinates, one easily gets

$$\tfrac{1}{2}\partial_{\alpha\beta}\left[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\right] = \kappa(-g)\,(t^{\mu\nu}_{LL} + T^{\mu\nu})\,. \tag{5.9}$$

First, after substituting from the Einstein equations $\kappa T^{\mu\nu} = G^{\mu\nu}$, Eq. (5.9) transfers to the identity. Thus, indeed it is equivalent to the Einstein equations. Second, one finds that $\sqrt{-\overline{g}}\hat{t}^{\mu\nu}_{(g-red)}$ is the covariantized Landau-Lifshitz's psedotensor $(-g)t^{\mu\nu}_{LL}$ [1].

Let us discuss gauge invariance properties of the systems presented by the field equations in the forms (3.2) and (5.8). The gauge transformations for the dynamical variables in the framework of the field formulation of GR on a flat background are defined in [8]:

$$\hat{l}'^{\mu\nu} = \hat{l}^{\mu\nu} + \sum_{k=1}^{\infty} \frac{1}{k!}\ £^k_\xi \left(\hat{\overline{g}}^{\mu\nu} + \hat{l}^{\mu\nu}\right), \qquad \phi'^A = \phi^A + \sum_{k=1}^{\infty} \frac{1}{k!}\ £^k_\xi \phi^A. \tag{5.10}$$

For the both the systems, the Lagrangians are gauge invariant up to divergences with taking into account $\overline{R}_{\mu\nu\alpha\beta} = 0$. The equations (3.2) and (5.8) are gauge invariant on themselves and with $\overline{R}_{\mu\nu\alpha\beta} = 0$. Concerning the energy-momentum tensors, both $\hat{t}^{(tot)}_{\mu\nu}$ and $\hat{t}^{(mod-tot-red)}_{\mu\nu}$ are not gauge invariant. Even on the dynamical equation one has

$$\kappa\hat{t}'^{(tot)}_{\mu\nu} = \kappa\hat{t}^{(tot)}_{\mu\nu} + \hat{G}^L_{\mu\nu}(l' - l), \qquad \kappa\hat{t}'^{(mod-tot-red)}_{\mu\nu} = \kappa\hat{t}^{(mod-tot-red)}_{\mu\nu} + \hat{G}^{L(mod)}_{\mu\nu}(l' - l). \tag{5.11}$$

The non-invariance is expressed through the covariant divergences as it follows from the form of the operators $\hat{G}^L_{\mu\nu}$ and $\hat{G}^{L(mod)}_{\mu\nu}$. It is not surprisely because Eq. (5.11) expresses the non-localization of the energy chracteristics in GR. Besides, the interpretation of the non-localization in the form (5.11) is useful because it is given numerically.

Again, we stress a big advantage of the first derivatives only in the energy-momentum tensor. However, second derivatives which appear in the energy-momentum tensor density (2.10) need some comments. In [20] we have shown that the symmetrized quantities constructed in [19] coincide with the ones presented here and included into the conservation law (3.11). Due to this, and using the dynamic and background equations one can conclude that zero's component $\hat{I}^0$ of the conserved current in (3.11), based on $\hat{t}^g_{\mu\nu}$, contains only the first *time* derivatives of $\hat{l}^{\mu\nu}$, therefore has the normal behaviour with respect to initial conditions with the definition of the quantities, like (3.6). This requirement, at least, may be unnecessarily restrictive.

6 Massive (finite-range) gravity

Babak and Grishchuk using their technique [21] have constructed the one interesting variant of the massive theory of gravity [22] that we give in the present section. It is natural to assume that the Lagrangian may also include an additional term similar to the one in Eq. (5.5), but where the quantity $\tilde{R}_{\alpha\rho\beta\sigma}$ is the curvature tensor of an abstract spacetime with a constant non-zero curvature: $\tilde{R}_{\alpha\rho\beta\sigma} = K\left(\tilde{g}_{\alpha\beta}\tilde{g}_{\rho\sigma} - \tilde{g}_{\alpha\sigma}\tilde{g}_{\rho\beta}\right)$ where K is with demensionality of $[length]^{-2}$. If one adds $\hat{\Lambda}^{\alpha\beta\rho\sigma}\tilde{R}_{\alpha\rho\beta\sigma}$ with $\hat{\Lambda}^{\mu\nu\alpha\beta} = (4\sqrt{-\overline{g}})^{-1}\left(\hat{l}^{\alpha\nu}\hat{l}^{\beta\mu} - \hat{l}^{\alpha\beta}\hat{l}^{\mu\nu}\right)$, changing $\tilde{g}^{\mu\nu} \to \overline{g}^{\mu\nu}$, then the additional term in the Lagrangian (5.5) is $\frac{1}{2}\sqrt{-\overline{g}}K\left(l^{\alpha\beta}l_{\alpha\beta} - l^\alpha_\alpha l^\beta_\beta\right)$. Clearly, the new theory is not GR, but one recognizes in this term the Fierz-Pauli mass-term [33]. Thus, noting that the structure (5.5) generates mass terms and finding that only two independent quadratic combinations of $l^{\mu\nu}$ exist, Babak and Gishchuk arrive at a 2-parametric family of theories with the additional mass terms in the gravitational Lagrangian (5.5):

$$\hat{\mathcal{L}}^g_{(mass)} = \hat{\mathcal{L}}^g_{(mod)} + \sqrt{-\overline{g}}\left[k_1 l^{\alpha\beta}l_{\alpha\beta} + k_2(l^\alpha_\alpha)^2\right], \tag{6.1}$$

k_1 and k_2 have demensionality of $[length]^{-2}$; here, we use as independent variables $l^{\mu\nu} = \hat{l}^{\mu\nu}/\sqrt{-\overline{g}}$.

Of course, these theories do not coincide with GR. The additional term in (6.1) gives a contribution both into the r.h.s. and into the l.h.s. of Eq. (5.8), and the equations of the massive gravity theories symbolically could be rewritten as

$$\hat{G}^{\mu\nu}_{L(mass)} = \kappa \hat{t}^{\mu\nu}_{(tot-mass)}\,. \tag{6.2}$$

These equations are, of course, covariant under coordinate transformations in a background flat spacetime. However, unlike Eq. (3.2) and Eq. (5.8), the new field equations (6.2) are not gauge invariant. By this one needs no to construct transformations, like (5.11). Thus, there is no a problem with a localization of $\hat{t}^{\mu\nu}_{(tot-mass)}$ — it is localized.

To have a direct comparison with GR effects it is more convenient to present Eq. (6.2) in the quite equivalent quasi-geometrical form:

$$G_{\mu\nu} + M_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.3}$$

where the massive term is

$$M_{\mu\nu} \equiv \left(2\delta^\alpha_\mu \delta^\beta_\nu - g^{\alpha\beta} g_{\mu\nu}\right)\left(k_1 l_{\alpha\beta} + k_2 \overline{g}_{\alpha\beta} l^\rho_\rho\right). \tag{6.4}$$

First, one has the Bianchi identity $D_\nu G^\nu_\mu \equiv 0$ in effective spacetime. Second, on the matter equations (2.3) one has $D_\nu T^\nu_\mu = 0$, as usual. Thus, applying the same diffentiation one arrives at the non-trivial consequences of Eq. (6.3):

$$D^\nu M_{\mu\nu} = 0. \tag{6.5}$$

Although these equations are merely the consequences of the full system (6.3), and therefore contain no new information, it proves convenient to use them instead of some members of the original set (6.3).

In order to give a physical interpretation of k_1 and k_2, following to the analysis by Ogievetsky and Polubarinov [34], and by van Dam and Veltman [35], one considers the linearization of the Eqs. (6.3):

$$\tfrac{1}{2}\left(\overline{D}_\rho \overline{D}^\rho l_{\mu\nu} + \overline{g}_{\mu\nu}\overline{D}_\rho \overline{D}_\sigma l^{\rho\sigma} - \overline{D}_\rho \overline{D}_\nu l_\mu{}^\rho - \overline{D}_\rho \overline{D}_\mu l_\nu{}^\rho\right) + 2k_1 l_{\mu\nu} - (k_1 + 2k_2)\overline{g}_{\mu\nu} l^\alpha_\alpha = 0. \tag{6.6}$$

The divergence if this equation is

$$\overline{D}_\nu \left[2k_1 l^{\mu\nu} - (k_1 + 2k_2)\overline{g}^{\mu\nu} l^\alpha_\alpha\right] = 0, \tag{6.7}$$

which is the linearized version of Eq. (6.5), and is a consequence of Eq. (6.6).

Consider the first case with $k_1 \neq k_2$. The full system (6.6) is equivalent to

$$\Box H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 0, \tag{6.8}$$

$$\Box l^\alpha_\alpha + \beta^2 l^\alpha_\alpha = 0, \tag{6.9}$$

together with Eq. (6.7). Here, $\Box \equiv \overline{g}^{\alpha\beta}\overline{D}_{\alpha\beta}$,

$$H^{\mu\nu} \equiv h^{\mu\nu} - \frac{k_1 + k_2}{3k_1}\overline{g}^{\mu\nu} l^\alpha_\alpha - \frac{k_1 + k_2}{6k_1^2}\overline{D}^{\mu\nu} l^\alpha_\alpha + \frac{k_1 + k_2}{12k_1^2}\overline{g}^{\mu\nu}\Box l^\alpha_\alpha \tag{6.10}$$

with $\overline{g}_{\mu\nu}H^{\mu\nu} = 0$ and $\overline{D}_\nu H^{\mu\nu} = 0$. Thus, parameters in the wave-like equations (6.8) and (6.9) are

$$\alpha^2 = 4k_1 \qquad \text{and} \qquad \beta^2 = -2k_1(k_1 + 4k_2)/(k_1 + k_2) \tag{6.11}$$

can be thought as inverse Compton wavelengths of the *spin-2* graviton with the mass $m_2 = \alpha\hbar/c$ associated with the field $H^{\mu\nu}$ and of *spin-0* graviton with mass $m_0 = \beta\hbar/c$ associated with the field l^α_α.

With studing the weak gravitational waves in the massive gravity one finds certain modifications of GR. Thus the spin-0 gravitaional waves, presented by the trace $l = l^{\mu\nu}\eta_{\mu\nu}$, and the polarization state of the spin-2 graviton presented by the spatial trace $H^{ik}\eta_{ik}$ both, unlike GR, become essential. They provide additional contributions to the energy-momentum flux carried by the gravitaional wave, and the extra components of motion of the test particles. However, gravitational wave solutions, their energy-momentum characteristics, and observational predictions of GR are fully recovered in the massless limit $\alpha \to 0$, $\beta \to 0$.

For the case with the mass term of Fierz-Pauli type, $k_1 + k_2 = 0$, that corresponds $\beta^2 \to \infty$ (see (6.11)), the full set of equations (6.6) is equivalent to

$$l^\alpha_\alpha = 0, \qquad \Box l^{\mu\nu} + 4k_1 l^{\mu\nu} = 0, \qquad \overline{D}_\nu l^{\mu\nu} = 0.$$

This case is interpreted as unacceptable [22]. Even in the limit $\alpha \to 0$, there remains a nonvanishing "common mode" motion of test particles in the plane of the wave front. The extra component of motion is accounted for the corresponding additional flux of energy from the sourse, typically, of the same order of magnitude as the GR flux. This, at least, is in a conflict with already available inderect gravitational-wave observations of binary pulsars [36]. Such theories probably have to be rejected.

In [22], the full non-linear equations (6.3) were analyzed from the point view of the black hole and the cosmological solutions. Thus, searching for static spherically-symmetric solutions in vacuum it is necessary to consider the equations:

$$G^0{}_0 + M^0{}_0 = 0, \qquad G^1{}_1 + M^1{}_1 = 0, \qquad G^2{}_2 + M^2{}_2 = 0 \tag{6.12}$$

where the indeces correspond to spherical coordinates of the background. Unlike GR, the last equation in (6.12) is not a consequence of the first two ones. However, for the more convenient analysis of the system (6.12) it is useful to consider the simple combination of Eqs. (6.5) instead of the last equation in (6.12). The consideration is simplified if $\alpha = \beta$, however all the qualitative conclusions remain valid for $\alpha \neq \beta$. Combining analitical and numerical technique Babak and Grishchuk have demonstrated that the solution of the massive theory is practically indistinquishable from that of GR for all R larger than $2M$, but smaller than $1/\alpha$, where R and M are the radial and mass parameters of the Schwarzschild solution. For R larger than $1/\alpha$ the solution takes the form of the Yukawa-type potentials; therefore they call this massive theory as finite-range gravity. The massive solution is also deviate strongly from that of GR in the vicinity of $R = 2M$ that is the location of the globally defined event horizon of the Schwarzschild black hole in GR. In the massive gravity the event horizon does not form at all, and the solution smoothly continues to the region $R < 2M$ and terminates at $R = 0$ where the curvature singularity develops. Since the αM can be extrimely small, the redshift of the photon emitted at $R = 2M$ can be extrimely large, but it remains finite in contrast with GR solutions. Infinite redshift is reached only at the singularity $R = 0$. In the astrophysical sense, all conclusions that rely specifically on the existence of the black hole even horizon, are likely to be abandoned. It is very remarkable and surprising that the phenomen of black hole should be so unstable with respect to the inclusion of the tiny mass-terms, whose Compton wavelenghts can exceed, say, the present-day Hubble radius.

It was also considered homogeneous isotropic solutions in the framework of the massive gravity. Matter sources were taken in the simplest form of a perfect fluid with a fixed equation of state. There are two independent field equations from the set (6.3), unlike GR where there is only one in the same case. First, if the mass of the *spin-0* graviton is zero, $\beta^2 = 0$, the cosmological solutions are exactly the same as those of GR, independently of the mass of the *spin-2* graviton, i.e., independently of the value of α^2. This result is expected due to the highest spatial symmetry: the *spin-2* degrees of freedom have no chance to reveal themselves. Then, for $\beta^2 \neq 0$ it was considered technically more simple case $4\beta^2 = \alpha^2$ which was studied in full details. Qualitive results are valid for $4\beta^2 \neq \alpha^2$. Again, combining analytical approximations and numerical calculations it was demonstrated that the massive solution has a long interval of evolution where it is practically indistinguishable from the Friedmann solution of GR. The deviation from GR are dramatic at very early times and very late times. The unlimited expansion is being replaced by a regular maximum of the scale factor, whereas the singulariry is being replaced by a regular minimum of the one. The smaller β, the higher maximum and the deeper minimum, i.e., the arbitrary small term in the Lagrangian (6.1) gives rise to the oscilatory behaviour of the cosmological scale factor.

Following the logic of interpretation that α^2 and β^2 define the masses, they are thought as positive. However, the general structure of the Lagrangian (6.1) does not imply this. Then, if one allows α^2 and β^2 to be negative, the late time evolution of the scale factor exibits an "accelerated expansion" that is similar to the one governed by a positive cosmological Λ-term. The development of this point could be useful in the light of the modern cosmological observational data [37].

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