# Soliton Dynamics: Classical Analogues of Relativity


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The present paper is the review of results, obtained by various authors, testifying that there are analogues of relativistic effects in the framework of the classical mechanics. Such effects were observed, in particular, at a motion of topological solitons (kinks, dislocations) in solids. These effects are described by the formulas, similar to the formulas of the special theory of relativity but they contain sound velocity instead of velocity of light. It is shown that there is a number of non-linear systems, enabling the supersonic motion of solitons. Such solitons are mechanical analogues of hypothetic tachyons. These analogies are a part of more general connection between the theory of solitons and field theories. We consider an analogy between the theory of dislocations and electrodynamics as an example.


## 1. Introduction

At first sight it would seem that the special theory of relativity cannot have the classical analogue. However, there are the classical particle-like objects, solitons [1], which are the solutions of the Lorentz-covariant equations. Similar to the relativistic particles, these solitons have a continuous and limited spectrum of velocities $0 < v < v_s$ where $v_s$ is a sound velocity. The movement of these solitons is accompanied by effects, similar to the relativistic ones. Among these effects are Lorentz reduction of width of a moving soliton, velocity dependence of soliton energy according to the Lorentz law [1] etc. The formulas, describing these effects, are similar to the formulas of the special theory of relativity but they contain sound velocity instead of velocity of light. The origin of these formulas is explained as follows. The effects of the special theory of relativity are based on finiteness of the velocity of information transfer. In the mechanics of deformable solids the information, concerning soliton motion, is transferred by sound waves. Therefore in the formulas of the soliton theory the sound velocity occupies the place of velocity of light.

This analogy takes place both for solitons in one-dimensional systems and for dislocations in three-dimensional continua. However the classical mechanics also demonstrates more complicated relativistic effects. There are two velocities of information transfer in isotropic solids: velocities of longitudinal and transverse sound waves. Besides there are the gradient non-linearity and dispersion in solids. As a result of these features the atomic equations of motion are not Lorentz-covariant ones. However, there are the effects caused by finiteness of sound velocities in these solids. We have the right to keep the name ‘relativistic’ for referring to these effects. Some non-Lorentz-covariant equations have the supersonic soliton solutions. These solitons are mechanical analogues of hypothetic tachyons (superluminal particles). The discussions which follow point out that their existence does not break the causality principle and other decrees of nature.

It is well known that any optical effect has an acoustic analogue. It is less known that this analogy has a continuation. There is an analogue of classical electrodynamics (i.e. theory, describing not only propagation of electromagnetic waves but also their interaction with charged particles) in the framework of the classical mechanics. Kosevich [2] has shown that such an analogue is the dynamical theory of dislocations, i.e. topological solitons in a crystal lattice. The dislocations correspond to electrical charges, and the fields of elastic deformations and mechanical stresses are analogues of an electromagnetic field. Later Musienko and Koptsik [3] had shown that the dynamical theory of dislocations can be formulated as the gauge theory. The ambiguity of potential of the dislocation elastic field (‘gauge freedom’) is connected to the ambiguity of displacements around a dislocation.

In order to prevent misunderstanding we would like to explain the meaning of the term ‘soliton’. The mathematicians say that soliton is the localized particle-like solution of integrable nonlinear equation having finite energy [4]. The physicists usually use more wider definition and say that soliton is the localized stationary or stationary on average perturbation of homogeneous or spatially periodic nonlinear medium [5]. We use this definition. Topological solitons are a special case because they have topological charges [6].

The aim of this paper is to summarize relativistic effects in the classical mechanics, to compare them to the analogous effects of the special theory of relativity, to discuss their origin, to show that there are the ‘mechanical’ relativistic effects which do not have analogues in the special theory of relativity up till now, in particular, supersonic solitons. It is natural to suppose that such solitons are analogues of tachyons.

## 2. Relativistic effects in dynamics of solitons in one-dimensional systems

Let us consider classical one-dimensional Frenkel–Kontorova model of a dislocation [1, 7]. The chain of particles of mass *m*, connected by linear springs of rigidity *k*, interacts with sinusoidal potential (fig.1).

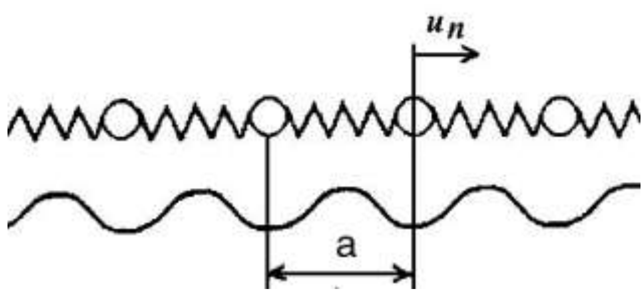


Fig. 1. One-dimensional Frenkel–Kontorova model.

The chain can be stretched (or compressed) in such a way that the number of particles is less (or more) than number of potential wells per unit. Such a configuration refers to as a kink (or, accordingly, antikink). Let $a$ is a period of potential, $U_n$ is a displacement of particle $n$ relative to well $n$, $u_n = U_n/a$. The equation of motion of the chain is

$$m\partial_t^2 u_n - k\left(u_{n+1} - 2u_n + u_{n-1}\right) + A\sin 2\pi u_n = 0$$

where $\partial_t \equiv \partial/\partial t$, $A$ is a constant, $-\infty<n<\infty$. In continuous approximation we obtain the sine-Gordon equation [1]

$$\frac{1}{c^2}\partial_t^2 u - \frac{1}{a^2}\partial_x^2 u + \frac{A}{k\,a^2}\sin 2\pi u = 0 \qquad (2.1)$$

where $c = a(k/m)^{1/2}$ is the sound velocity in the chain. We designate continuous spatial variable through $x$. Let $a = 1$, $A = 1$, $k = 1$. Then the kink (topological soliton) solution of the eq. (2.1) is (fig. 2)

$$u(x,t) = \frac{2}{\pi}\operatorname{atan}\exp\frac{x - vt}{\gamma} \qquad (2.2)$$

where $\gamma = (1 - v^2/c^2)^{1/2}$, $v$ is the kink velocity.

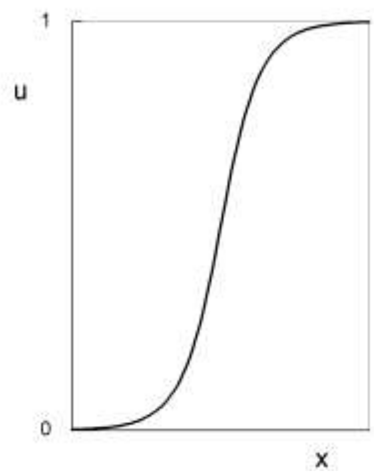


Fig. 2. Soliton (kink) solution of the sine-Gordon equation.

From formula (2.2) we conclude that the significant part of increasing $u$ occurs in the narrow region near the kink center. The width of this region is

$$L = \left(1 - \frac{v^2}{c^2}\right)^{1/2}. \qquad (2.3)$$

This is the width of the kink. It depends on the kink velocity according to the Lorentz law. However formula (2.3) contains the sound velocity instead of velocity of light.

The kinetic energy of the chain is

$$T = \frac{m}{2}\sum_n \left(\partial_t u_n\right)^2. \qquad (2.4)$$

The potential energy of the chain is

$$U=\frac{k}{2}a^2\sum_n\left(u_{n+1}-u_n\right)^2+\frac{Aa^2}{2\pi}\sum_n\left(1-\cos 2\pi u_n\right) \qquad (2.5)$$

The total energy of the kink is $E = T + U$. In continuous approximation the sums in (2.4-2.5) are replaced by integrals. Inserting the solution (2.2) in these integrals we obtain [1]

$$E=\frac{E_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad (2.6)$$

where $E_0$ is the kink rest energy. From formula (2.6) follows that if the kink velocity $v$ is significantly less than the sound velocity $c$ then kink kinetic energy $T=Mv^2/2$ where $M$ is the kink rest mass. It is connected to the kink rest energy $E_0$: $E_0 = Mc^2$.

The interactions of atoms in real solids are more complicated than the sinusoidal one. Their account results in breaking Lorentz symmetry of atomic equations of motion. Solutions of such equations are supersonic solitons. Toda [8] pioneered in analytical investigation of supersonic dynamical solitons in one-dimensional lattices. He considered the chain of particles with exponential interaction (the Toda lattice). Let $\rho_n \equiv u_{n-1} - u_n$ is a reduction of distances between the neighboring particles of mass $m$, caused by their displacements $u_n$. The equations of motion of particles is

$$Mb\partial_t^2\rho_n=k\left[\exp\left(b\,\rho_{n+1}\right)+\exp\left(b\,\rho_{n-1}\right)-2\exp\left(b\,\rho_n\right)\right]$$

The soliton solution of these equations is (fig. 3)

$$\rho_n=\frac{1}{b}\ln\left\{1+\frac{\sinh^2(qa)}{\cosh^2[q(na-vt)]}\right\} \qquad (2.7)$$

where the soliton velocity

$$v=\frac{c}{qa}\,\mathrm{sh}\left(qa\right), \qquad (2.8)$$

$c = a(k/M)^{1/2}$ is velocity of longitudinal sound waves in a harmonic chain, i.e. at $b = 0$. Parameter $q$ characterizes the reciprocal soliton width $q \approx 2\pi/L$ where $L$ is the soliton width. From formula (2.8) follows that the acoustic solitons (2.7) are always supersonic ones. If $v \rightarrow c$ then the soliton amplitude tends to zero and its width goes to infinity. Thus, such a soliton can not overcome the sonic barrier in this chain. These solitons are born with supersonic velocities. It is interesting to note that the similar behaviour was predicted for tachyons [9].

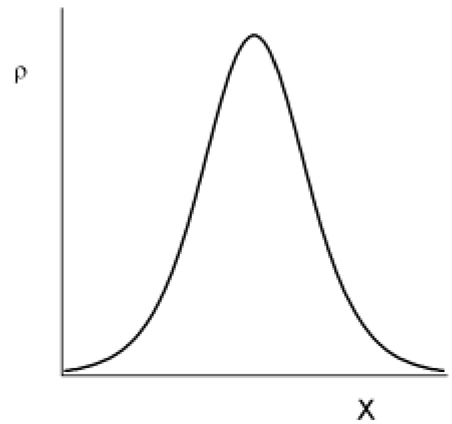


Fig. 3. Supersonic dynamical soliton in Toda lattice.

If the soliton width $L$ is much above the lattice parameter then we can use the continuous approximation. Then the solution (2.7) looks like

$$\rho\,(x,t) = \frac{1}{b}\left[\frac{\mathrm{sh}(qa)}{\mathrm{ch}(q\zeta)}\right]^2$$

where $\zeta = x - vt$. The soliton energy is $E = T + U$ where the kinetic energy is

$$T = \frac{M}{2a}\int\left[\partial_t u(\zeta)\right]^2 d\zeta = \frac{2Mv^2\,\mathrm{sh}^4(qa)}{3qa^3b^2},$$

the potential energy is

$$U = \frac{2k\,\mathrm{sh}^4(qa)}{3qab^2}\left[1 + \frac{4}{15}\mathrm{sh}^2(qa)\right].$$

If the velocity of the supersonic soliton goes to the sound velocity then its energy tends to zero. Thus, the properties of the Toda solitons in the supersonic region are opposite to the properties of the soliton solutions of Lorentz-covariant equations in the subsonic region where the energy of the soliton tends to infinity if its velocity goes to sound velocity and the soliton width tends to zero. In addition to the Toda soliton, other non-topological solitons in one-dimensional chains have the similar properties. The question arises as to whether there are the solitons, unifying these two extremes? It turns out that appropriate one-dimensional models exist. In order to construct such a model it is necessary to insert a non-linear chain, enabling dynamical solitons, in an external periodic field (the substrate field). Then these solitons acquire the topological charge. In such a case particles at the chain ends are in minima of the substrate potential. If the interaction with the substrate is rather weak then the solitons can keep the 'generic' property of dynamical solitons that is supersonic motion.

## 3. Supersonic topological solitons

Kosevich and Kovalev [10] pioneered in investigation of supersonic topological solitons. They considered an anharmonic chain of particles in sinusoidal potential. In continuous approximation the equation of motion of the chain is

$$\partial_t^2 u - c^2\left[\partial_x^2 u + \frac{a^2}{12}\partial_x^4 u + \frac{\pi^2}{2}\left(\partial_x u\right)^2 \partial_x^2 u\right] + \frac{2\pi}{am} U \sin\frac{2\pi u}{a} = 0 \qquad (3.1)$$

where *m, c, a, U* are constants. The solution of this equation is a kink, having velocity *v*,

$$u = \frac{2a}{\pi}\,\mathrm{atan}\exp\left(-\frac{x - vt}{L_K}\right) \qquad (3.2)$$

where the kink width $L_K$ (curve 3 in fig. 4) is the solution of the biquadratic equation

$$48\pi^2 U L_K^4 + 12ma^2(v^2 - c^2)\, L_K^2 - mc^2 a^4 = 0 \qquad (3.3)$$

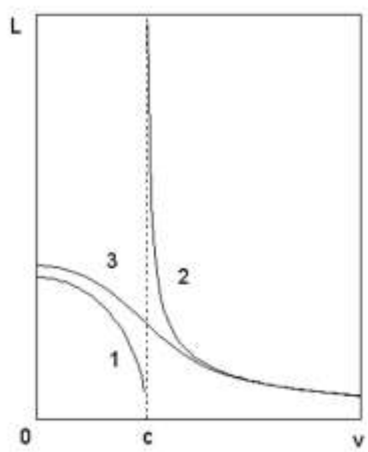


Fig. 4. The widths of various solitons *L* vs. their velocity *v*: 1 – for the sine-Gordon kink (the Lorentz law); 2 – for the supersonic dynamical solitons; 3 – for the Kosevich–Kovalev soliton.

If the interaction of the chain with the substrate is stronger than the interaction between the particles, i.e. U » $mc^2$, then the last item in the left side of eq. (3.3) can be neglected. In such a case the soliton width–velocity relationship has the Lorentz form (curve 1 on fig. 4)

$$L_L = \frac{ac}{2\pi}\sqrt{\frac{m}{U}\left(1 - \frac{v^2}{c^2}\right)}.$$

The second extreme case is a free chain without a substrate ($U$=0). Then the soliton width–velocity relationship has the form which is typical of supersonic dynamical solitons (curve 2 on fig. 4)

$$L_D = \frac{a}{2\sqrt{3\left(\frac{v^2}{c^2}-1\right)}}.$$

Thus, the function $L_K(v)$ for the Kosevich–Kovalev topological soliton is ‘matching’ two solutions: the Lorentz function $L_L(v)$ for subsonic topological solitons and the function $L_D(v)$ for supersonic dynamical solitons. Soliton (3.2) has a continuous spectrum of velocities in the range from zero to infinity. The sound velocity is not the singular point for this soliton.

We would like to note that now there is not tachyonic theory which would admit the similar ‘desingularization’ near the velocity of light. The question arises as to whether there are the analogues of soliton (3.2) in the non-linear field theories? If the velocity of such a soliton is small then it should behave like a usual particle following Lorentz laws. However if its velocity will verge towards the velocity of light then the deviation from the Lorentz behaviour should increase. At last the soliton would get the superluminal velocity. During the passage soliton velocity through the velocity of light any soliton characteristics would not approach zero or infinity.

Kosevich and Kovalev [10] also have considered the similar model using another potential of particles’ interaction. Potential of interaction with the substrate was chosen in the polynomial form. Then the equation of motion of the chain is

$$\partial_t^2 u - c^2\left[\partial_x^2 u + \frac{a^2}{12}\partial_x^4 u - 3\beta\,\partial_x u\,\partial_x^2 u\right] + \frac{2U}{m\,a^4}u(a-u)(a-2u) = 0. \tag{3.4}$$

This equation describes, in particular, dynamics of a bistable molecular chain. Equation (3.4) has the soliton solution

$$u = \frac{a}{1+\exp\frac{3\beta(x-vt)}{a}} \tag{3.5}$$

This soliton can move at the single velocity

$$v = c\sqrt{1+\frac{3\beta^2}{4}-\frac{2U}{9mc^2\beta^2}}$$

If $\beta^4 > 8U/(27mc^2)$ then this velocity is supersonic one. The discrete spectrum of velocities is the general property of many supersonic topological solitons.

Later this model was investigated by Savin [11] who modified the substrate potential. Then equation (3.4) has changed into the following

$$\left(1-v^2\right)\partial_\zeta^2 u+\frac{1}{12}\partial_\zeta^4 u-3\beta\,\partial_\zeta u\,\partial_\zeta^2 u-4Gu\left(u^2-1\right)=0 \tag{3.6}$$

where $\zeta = x - vt$, $a = 1$, $c = 1$, $m=1$. If the substrate is absent ($G = 0$) then equation (3.6) evolves into the Boussinesq equation after an integration. Its solution is a supersonic dynamical soliton

$$\varphi=\frac{1-v^2}{\beta}\operatorname{sech}^2(q\zeta)$$

where $v$ is the soliton velocity, $\varphi\equiv\partial_\zeta u$. In the presence of the substrate this soliton has the topological charge

$$Q(v)=u(+\infty)-u(-\infty)=\int_{-\infty}^{+\infty}\varphi(\zeta)\,d\zeta=-\left[\frac{4\left(v^2-1\right)}{3\beta^2}\right]^{1/2}=-1\,. \tag{3.7}$$

If the chain contains $N$ identical dynamical solitons then equation (3.7) looks like $NQ(v_N)= -1$. Then the velocity of each soliton

$$v_N=\left[1+\frac{3}{4}\left(\frac{\beta}{N}\right)^2\right]^{1/2}.$$

Thus, at the limit $G \rightarrow 0$ the topological solitons of this model have limited discrete supersonic spectrum of velocities. The sound velocity is the accumulation point of this spectrum. Certainly, the supersonic kink can move at other velocities which are absent in the discrete spectrum. The numerical modeling has shown [11] that in such a case the kink radiates sound waves until it reaches the nearest velocity from the discrete spectrum. Then the radiation ceases.

It is worth the reader's attention to observe that the discussed supersonic solitons differ essentially from the superluminal electromagnetic solitons [12]. The latter propagate in non-equilibrium media only and do not transfer information. Unlike them, the supersonic topological solitons propagate in equilibrium systems and transfer information. The study of these solitons may be useful for the solution of the tachyonic problem. It is widely believed that it is impossible to observe the tachyons in the real world because their existence should result in paradoxes: breaking the causality principle, imaginary energy of particles etc. The considered analogy allows to make some hypotheses concerning tachyons. Really, the theory of solitons in solids is similar to the special theory of relativity in the subsonic region. Any relativistic effect has an analogue in the theory of solitons. This statement is true both for one-dimensional systems and for three-dimensional ones. However in the theory of solitons in solids we can repeat the usual arguments against an existence of tachyons. Really, the absolute value of parameter $c$ in the Lorentz root is not important for these arguments: $c$ can be equal to the velocity of light or the sound velocity. These arguments result in the conclusion that the supersonic solitons can not exist. However the considered examples of such solitons contradict this conclusion. Analytical studies and numerical modeling of supersonic solitons show that the supersonic movement do not lead to appearance of imaginary soliton energy and breaking the causality principle. The explanation of this contradiction is simple. Near the sound velocity and in the supersonic region the soliton equations are not Lorentz-covariant. Hence, they can have the supersonic solutions. The standard anti-tachyonic arguments are unfounded in such a case as they are based on using the Lorentz transformations in the supersonic region where these transformations are inapplicable.

## 4. Relativistic effects in the dislocation dynamics

Let us consider a straight screw dislocation in a three-dimensional crystal. Dislocation is parallel to an axis *z* and move at velocity *v* along an axis x (fig. 5).

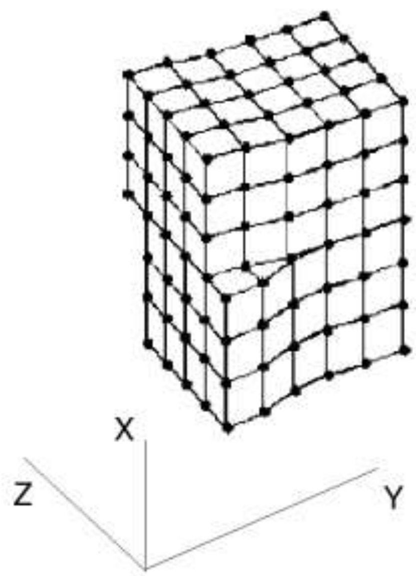


Fig. 5. The screw dislocation in a three-dimensional crystal.

Let us use continuous approximation and consider isotropic and linear elastic medium. Then the displacements of medium points $u_3$ around the dislocation are the solutions of the equation

$$\mu\left(\partial_1^2 + \partial_2^2\right)u_3 = \rho\,\partial_t^2\,u_3 \tag{4.1}$$

on condition that

$$\oint_L du_3 = b.$$

Here $\mu$ is the shear modulus, $\rho$ is the density of the medium, $b$ is magnitude of the Burgers vector. The dislocation solution of equation (4.1) is [7]

$$u_3 = \frac{b}{2\pi} \operatorname{atan} \frac{\gamma y}{x - vt} \tag{4.2}$$

where $\gamma=(1 - v^2/c_t^2)^{1/2}$, $c_t = (\mu/\rho)^{1/2}$. The point of origin is situated on the dislocation. The displacements $u_1$ and $u_2$ are equal to zero.

Let us use the cylindrical co-ordinates. The single non-zero component of mechanical stress tensor, induced by the dislocation, is

$$\sigma_{\varphi z}(r,\varphi) = \frac{\mu b \gamma}{2\pi r\left(\cos^2\varphi + \gamma^2 \sin^2\varphi\right)} \tag{4.3}$$

where $r^2 = (x - vt)^2 + y^2$, an angle $\varphi$ is counted off an axis $x$. The electric field strength of an infinite straight charged rod, parallel to an axis $z$ and moving at velocity $v$ along an axis $x$, has the same dependence on co-ordinates and velocity.

The kinetic energy of the screw dislocation

$$T = \frac{\rho}{2} \int (\partial_t u_3)^2 dV = \frac{E_0 v^2}{2\gamma c_t^2}$$

where the rest energy of the dislocation

$$E_0 = \frac{\mu b^2}{4\pi} \ln \frac{R}{r_0},$$

$R$ and $r_0$ are limits of integration over $r$. As a rule $R$ is equal to the distance apart the dislocation and crystal surface, $r_0$ is the lattice parameter. Potential energy of the screw dislocation

$$U = \frac{1}{2} c_{idfh} \int \partial_d u_i \partial_h u_f \, dV = \frac{E_0}{\gamma}\left(1 - \frac{v^2}{2c_t^2}\right)$$

where $c_{idfh}$ is the tensor of elastic modules. The total energy of the screw dislocation

$$E = T + U = \frac{E_0}{\left(1 - \frac{v^2}{c_t^2}\right)^{1/2}}$$

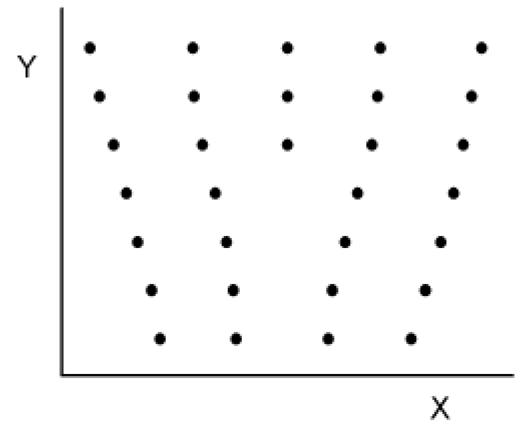


Fig. 6. An edge dislocation in a two-dimensional crystal.

Let us consider an edge dislocation in a two-dimensional crystal (fig. 6). Let us use continuous approximation and consider isotropic and linear elastic medium. Then the displacements of medium points around the edge dislocation, having the Burgers vector $\boldsymbol{b}$ = ($b$, 0) and moving at the velocity $\boldsymbol{v}$ = ($v$, 0), are the solutions of the system of equations [7]

$$\mu \partial_j \partial_j u_i + (\mu + \lambda) \partial_i \partial_j u_j = \rho \partial_t^2 u_i \tag{4.4}$$

on condition that

$$\oint_L du_i = b.$$

Here $\lambda$ is a Lame constant. There are longitudinal and transverse sound waves in the two-dimensional crystal. Their speeds never coincide. Therefore the formulas of dislocation dynamics are not Lorentz-covariant ones. The dislocation solution of equations (4.4) is [13]

$$u_1(x,y,t) = \frac{bc_t^2}{\pi v^2}\left[\operatorname{atan}\frac{y\left(1-\frac{v^2}{c_l^2}\right)^{1/2}}{x-vt} + \left(\frac{v^2}{2c_t^2}-1\right)\operatorname{atan}\frac{y\left(1-\frac{v^2}{c_t^2}\right)^{1/2}}{x-vt}\right],$$

$$u_2(x,y,t) = \frac{bc_t^2}{2\pi v^2}\left\{\frac{\frac{v^2}{2c_t^2}-1}{\left(1-\frac{v^2}{c_t^2}\right)^{1/2}}\ln\left[(x-vt)^2+\left(1-\frac{v^2}{c_t^2}\right)y^2\right] + \left(1-\frac{v^2}{c_l^2}\right)^{1/2}\ln\left[(x-vt)^2+\left(1-\frac{v^2}{c_l^2}\right)y^2\right]\right\} \tag{4.5}$$

where $c_l = [(\lambda+2\mu)/\rho]^{1/2}$ is the speed of longitudinal sound waves. The point of origin is situated on the dislocation. The mass $M$ and energy $E_0$ of an edge dislocation are related by the formula [13]

$$E_0 = \frac{Mc_t^2}{1+\frac{c_t^4}{c_l^4}}.$$

Thus, in the two-dimensional soliton theory Lorentz transformations are not the universal tool for calculation of the soliton characteristics. The overwhelming majority of physicists suppose that it is impossible to construct the relativistic non-Lorentz-invariant theory. But such a theory already exists and has experimental confirmations. It is the dynamical theory of dislocations. There are analogues of any effect of the special theory of relativity in the framework of this dislocation theory. Thus, we can use the term 'relativistic' for this theory. However the dynamical theory of dislocations transforms into the Lorentz-invariant one at the limit $c_l \rightarrow \infty$.

All these results are correct for a straight edge dislocation in the three-dimensional crystal. Its Burgers vector is normal to a dislocation line. The Burgers vector of a screw dislocation is parallel to this line. In the general case the angle this vector makes with the dislocation is not equal neither 0 nor $90^0$. The dislocation can be edge and screw at different points of its line. The theory of screw dislocations is Lorentz-invariant and the theory of edge dislocations is not Lorentz-invariant. Hence, there is a continuous transition from Lorentz-covariant formulas to non-Lorentz-covariant ones. This transition follows from the general dislocation theory. So, formulas (4.2, 4.5) can be obtained from the Mura formulas [14] for the dislocation distortions. These relations look much simpler if we use four-dimensional co-ordinates [3]

$$\beta_{jn}(x_f) \equiv \partial_n u_j(x_f) = \frac{C_{iabd}}{c_t^2} e_{nhag} \int_{\Omega} J^h{}_i{}^g(x'_f) \partial_d G_{bj}(x_f - x'_f) d\Omega' \qquad (4.6)$$

where $b, i, j$ = 1, 2, 3; $a, d, f, g, h, n$ = 0, 1, 2, 3; $c_t = (c^{1212}/\rho)^{1/2}$, $\partial_0 \equiv c_t^{-1}\partial_t$,

$$C_{iabd} = \begin{cases} c_{iabd} & \text{if } a = 1,2,3; \\ -\delta^{bi}\delta^{ad}\rho c_t^2 & \text{if } a = 0, \end{cases}$$

$c^{iabd}$ is the three-dimensional tensor of elastic modules, $\delta^{bi}$ is a Kroneker symbol, $e_{nhag}$ is an antisymmetric Levi-Civita tensor, $e^{0123} = 1$,

$$J^h{}_i{}^g(x'_f) = \tau^h b_i V^g \delta(x'_f - x^0_f)$$

is the tensor of dislocation flux density, $\tau^h$ is a vector tangent to the dislocation line, $V^g = (c_t, -\boldsymbol{V})$ is a four-dimensional dislocation velocity, $\boldsymbol{V}$ is a three-dimensional vector of dislocation velocity, $\delta(x'_f - x^0_f)$ is the delta function, $x^0_f$ are co-ordinates of the dislocation line, $d\Omega' = dV'd(c_t t')$ is a four-dimensional differential of spatio-temporal volume, $G_{bj}$ is the tensorial Green function of the equations of the classical linear theory of elasticity.

The listed relativistic effects are consequences of signal delay in soliton motion. It appears from this that only the physical quantities, connected to the signal delay in soliton motion: dislocation fields of elastic deformations and mechanical stresses, soliton energy etc., depend on the soliton velocity through Lorentz (or another relativistic) law. All other quantities, which are not connected to the signal delay (for example, lattice parameters), do not depend on the soliton velocity.

## 5. The gauge theory of dislocations and electrodynamics

We already mentioned the equivalence (with the only difference in constants) between the formula, describing dependence of mechanical stresses, induced by a straight screw dislocation, on its velocity and co-ordinates, and the formula, describing dependence of electric field strength of an infinite straight charged rod on its velocity and co-ordinates. It is obvious that this similarity is not a random coincidence. It is a part of general analogy between the dynamic theory of dislocations and electrodynamics [2, 3]. It is necessary to understand this analogy to gain a better insight into interconnections between relativistic effects in electrodynamics and classical mechanics. We consider just electrodynamics in this paper because it is the well-known field theory. From the general point of view the dynamic theories of topological solitons are special cases of gauge field theories.

Evidently dislocations are sources of an elastic field (field of mechanical stresses and deformations), and electrical charges are sources of an electromagnetic field. However any defects in crystals, for example, cracks, are sources of an elastic field. The distinctive feature of line defects (dislocations and disclinations) is their topological character. It is primary cause of gauge nature of the dynamic theory of these defects. Topological charges of defects (Burgers vectors for dislocations and Frank vectors for disclinations) are similar to electrical charges. These charges are conserved quantities. The gauge transformations in the theory of dislocations are analogues of well-known gradient transformations of potential in electrodynamics. Let us consider a straight screw dislocation parallel to axis $z$ (fig. 5) as an example. We use cylindrical co-ordinates, point of origin is on a dislocation. In continuum approximation (we consider isotropic continuum) field of particles displacements in medium around a static dislocation looks like $u_z = b\ \varphi/2\pi$. We can measure angle $\varphi$ off any line normal to dislocation. Transition from one of these lines to another (i.e. the transition to another system of co-ordinates) results in addition of constant to displacements $u_z$. Distortions $\beta_{jn} = \partial_n u_j$ and the mechanical stresses

$$\sigma_{bh} = C_{bhjn}\,\beta_{jn}$$

do not change after such a transition. The electrodynamical analogy of this procedure is gauge transformation of potential $A_j$, conserving tensor of an electromagnetic field $F_{jn} = \partial_j A_n - \partial_n A_j$ (components of this tensor are vectors of electrical field strength $\boldsymbol{E}$ and magnetic field strength $\boldsymbol{H}$).

In order to prove that just tensors of distortions $\beta_{jn}$ and mechanical stresses $\sigma_{jn}$ are analogues of tensors of an electromagnetic field $F_{jn}$ and $H_{jn}$ let us compare density of energy of an electromagnetic field and density of energy of an elastic field. Density of energy of an electromagnetic field has form

$$W_{em} = \frac{1}{16\pi}\left(-4F^{0i}H^{0}{}_{i} + F_{in}H^{in}\right) = \frac{1}{8\pi}\,(\boldsymbol{ED} + \boldsymbol{HB})$$

where $i$, $n$ = 1, 2, 3, $\boldsymbol{D}$ is an electric induction, $\boldsymbol{B}$ is a magnetic induction. Density of energy of an elastic field has the following form in three-dimensional notation

$$W_{el} = \frac{1}{2}\left[\rho\left(\partial_t u_i\right)^2 + \beta_{in}\sigma_{in}\right].$$

Using 4-dimensional notation we can write a density of elastic field energy in the following form

$$W_{el} = \frac{1}{2}\left(-\beta^{i0}\sigma_i{}^0 + \sigma_{in}\beta^{in}\right).$$

It is necessary to note the important distinction between the gauge theory of dislocations and electrodynamics. In electrodynamics just as in Yang–Mills gauge theories tensor of a gauge field $F^a{}_{jn}$ (in electrodynamics $a$ = 1) is antisymmetric one relative to permutation of indices $j$ and $n$. In the theory of elasticity analogous tensors $\beta_{jn}$ and $\sigma_{jn}$ are not neither symmetric, nor antisymmetric in the general case. In the framework of the linear theory of elasticity it is usual to neglect an antisymmetric part of tensor of distortions, describing small rotations of continuum, and antisymmetric part of tensor of mechanical stresses (so called torque). Then we can use symmetric tensor of deformations $\varepsilon_{jn} = (\partial_n u_j + \partial_j u_n)/2$ instead of tensor of distortions. Now it is obvious that the tensor of deformations $\varepsilon_{jn}$ is an analogue of the tensor of an electromagnetic field $F_{jn}$.

Different symmetries and ranks of tensors, describing gauge fields, imply that it is impossible to use unmodified formulas of electrodynamics or Yang–Mills gauge theories in the gauge theory of dislocations. For example, Lagrangian of interaction of dislocations with elastic fields cannot have the form of convolution of a vector of particles displacements in a medium $u_i$ and tensor of dislocations flux $J\,{}^{hg}_{i}$ because the latter is a tensor of the third rank. Further we briefly describe a method of construction of Lagrangian of interaction of dislocations and elastic fields, proposed by Musienko and Koptsik [3].

Using the above 4-dimensional notation we can write Lagrangian of elastic fields in anisotropic continuum in the form

$$L_0 = -\frac{1}{2}C^{irjn}\,\partial_r u_i \partial_n u_j \,. \tag{5.1}$$

Here $n$, $r$ = 0, 1, 2, 3; $i$, $j$ = 1, 2, 3. We substitute each distortion in Lagrangian (5.1) by the sum of distortions induced by an elastic field and distortions (4.6) generated by dislocations. After certain transformations of the result obtained, we arrive at a Lagrangian that describes interaction between elastic fields and dislocations

$$L_{\text{int}} = -\frac{1}{c_t}B_{gij}\,K^{gij}$$

where the 4-dimensional tensor of the dislocations flux is

$$K^{gij} = e^{gajb} J_{a\ b}^{\ i} ,$$

$a, i$ = 1, 2, 3; $b, g, j$ = 0, 1, 2, 3; $B_{gij}$ is a tensorial potential which is an analogue of vector potential $A_j$ in electrodynamics. The 4-dimensional tensor of mechanical stresses is related to the tensorial potential as

$$\sigma_{ij} = \partial^d B_{dij} .$$

The complete Lagrangian of the medium with dislocations is

$$L = L_0 + L_{int} + L_m . \tag{5.2}$$

Here $L_m$ is the material Lagrangian, describing the energy of dislocations without taking their elastic interactions into account. Its electrodynamical analogue is Dirac Lagrangian of electron-positron field. An exact form of this Lagrangian for dislocations has not been determined as yet. For this reason it is not possible to construct Lagrangian of interaction of dislocations and elastic fields, using canonical method. This method consists in replacement of partial derivatives in material Lagrangian on covariant ones. Some authors (for example, Kadić and Edelen [15]) applied this method but they used quite another analogy between the gauge theory of line defects and field theories (in particular, electrodynamics). In their approach dislocations flux is an analogue of electrical and magnetic fields stregth and induction. Material Lagrangian in this theory is Lagrangian of an elastic field in free of defects continuum. Kadić and Edelen described interaction of defects with elastic fields, replacing partial derivatives in Lagrangian of an elastic field on covariant ones. We suppose that this approach is incorrect one. We shall show that topological defects are analogues of charged particles. However Kadić and Edelen [15] assumed that these defects are analogues of uncharged and massless quanta. Elastic field is an analogue of an electromagnetic field in our approach, as well as in paper by Kosevich [2].

The analogue of Dirac Lagrangian was found for sine-Gordon model. Coleman [16] found bosonization relations, establishing equivalence between sine-Gordon equation and Thirring equation which is Dirac equation with 4-fermion interaction. However dislocations, as well as sine-Gordon kinks, are topological solitons. Moreover, edge dislocations are many-dimensional generalizations of sine-Gordon kinks. So, we can hope that there is an analogue of bosonization relations for dislocations. But this is open problem until now.

Varying the field potentials, we obtain the following equations for the elastic field from (5.2)

$$\partial^h \sigma_{nh} = \frac{1}{c_t} C_{ngij} K^{gij} \tag{5.3}$$

where $i, n$ = 1, 2, 3; $g, h, j$ = 0, 1, 2, 3. Kosevich [2] found these equations for static dislocations. Equations (5.3) are analogues of the second pair of Maxwell equations

$$\partial_h H^{nh} = -\frac{4\pi}{c} j^n \tag{5.4}$$

There is also an analogue of the first pair of Maxwell equations in the theory of dislocations:

$$\partial_g \partial_a u_n - \partial_a \partial_g u_n = e_{giah} J^{i\,h}_{\ n} . \qquad (5.5)$$

Here $i, n$ = 1, 2, 3; $a, g, h$ = 0, 1, 2, 3. These equations are definition of a dislocation and statement about absence of disclinations in the continuum. They are similar to the first pair of Maxwell equations

$$e^{jglb} \partial_g F_{lb} = 0. \qquad (5.6)$$

It is well known that these equations are the statement about absence of magnetic charges (Dirac monopoles) in nature. The equations (5.3), as well as their analogues (5.4), are obtained by variation of Lagrangian. The equations (5.5) and (5.6) are not variational ones.

Using the principle of the least action and Lagrangian (5.2), we determine the force, acting on the unit length of line defect from the side of the elastic field:

$$f_i = \frac{1}{c_t} \int_V K_{idj} \sigma^{dj} dV . \qquad (5.7)$$

For a static dislocation the expression (5.7) have the following form in three-dimensional notation:

$$f_i = e_{igt} \tau_g b_d \sigma_{dt} . \qquad (5.8)$$

It is Peach–Kohler force, analogue of Coulomb force in electrodynamics . Let us consider, for example, two parallel screw dislocations in isotropic medium with Burgers vectors $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ parallel to axis $z$. Let us substitute mechanical stress (4.3), induced by one of dislocations, in the right side of formula (5.8). Then we obtain expression for a force of dislocations interaction in cylindrical co-ordinates

$$\boldsymbol{F} = \frac{\mu(\boldsymbol{b_1 b_2})\boldsymbol{r}}{2\pi r^2}$$

where $\boldsymbol{r}$ is vector, connecting two dislocations. Hence, likely charged dislocations ($\boldsymbol{b_1}\,\boldsymbol{b_2} > 0$) repel, and oppositely charged ones ($\boldsymbol{b_1}\,\boldsymbol{b_2} < 0$) attract.

The dynamical part of force (5.7) has the following form in three-dimensional notation:

$$f_i = \rho\, v_n b_n e_{irl} \tau_r V_l .$$

Here $v_n$ is the velocity of medium particles. It is so-called dislocation Lorentz force, obtained by Kosevich [2] in a different way.

Mensky [17] proved that the gauge group of the theory is a representation of fundamental group of order parameter space. Order parameter in the theory of dislocations and disclinations is displacement of medium particles. Then the fundamental group is SO(3) λ T(3), where λ is a symbol of semidirect product of groups of rotations SO(3) and translations T(3) in three-dimensional space. Hence, the gauge group of the dynamical theory of line defects is SO(3) λ T(3) in the most general case.

There is a widespread opinion (see, for example, [18]) that the conservation of topological charges is not connected to symmetries of Lagrangian of the system. In other words, the conservation law of topological charges is not the consequence of the Noether theorem. However Musienko and Koptsik [3] proved that the conservation laws of topological charges of dislocations and disclinations results from the second Noether theorem in the framework of the gauge theory of line defects. Let us consider following gauge transformations of tensorial potential, conserving observable fields (mechanical stresses and deformations):

$$B_{\gamma i\beta} \rightarrow B`_{\gamma i\beta} = B_{\gamma i\beta} + e_{\gamma\beta\delta\alpha}\, \partial^{\alpha}\, \varepsilon_i^{\ \delta} \qquad (5.9)$$

Here $i$ = 1, 2, 3, the Greek indices varies from 0 up to 3, $\varepsilon_i^{\ \delta}$ is arbitrary tensorial function of co-ordinates and time. According to the second Noether theorem there is a conservation law, corresponding to every gauge transformation. Transformations (5.9) correspond to the conservation laws

$$\partial_{\mu}\, (J^{\mu\ \nu}_{\ i} - J^{\nu\ \mu}_{\ i}) = 0. \qquad (5.10)$$

It is an analogue of the conservation law of electric charge in electrodynamics which is also consequence of the second Noether theorem. At $\nu = 0$ formula (5.10) means that dislocations cannot terminate inside a crystal. They form closed loops or terminate on crystal surface. If $\nu \neq 0$ then this formula transforms into the continuity equation for dislocations flux.

## 6. Conclusions

The examined analogy between the special theory of relativity and soliton dynamics allows us to revise the conventional opinions on an origin of relativistic effects. Contrary to prevailing view the analogues of all effects of the special theory of relativity can be obtained in the framework of the classical Newtonian mechanics. Relativistic effects in soliton theory are much more complicated than their electrodynamical analogues. The question arises as to whether it is possible to continue analogy between the special theory of relativity and soliton theory in the superluminal region. This issue remains open up till now. However this analogy allows us to advance a hypothesis that there are superluminal solitons in the framework of the nonlinear field theories.

The analogy between relativistic effects in the theory of solitons in solids and the special theory of relativity is closely connected to another analogy: between the gauge theory of dislocations and electrodynamics. All these analogies are arguments for the soliton models of elementary particles. Such models are possible in the framework of nonlinear field theories. The contradiction between nonlinear interaction of fields and Lorentz symmetry was always insurmountable obstacle for construction of nonlinear field theories. Considered examples demonstrate solution of this contradiction in condensed matter physics. The theory of solitons in solids, not satisfying to the condition of Lorentz symmetry, is a promising model for construction of the generalized special theory of relativity.

It is considered to be the case that the finiteness of speed of information transfer imply Lorentz symmetry of physical theories. The previous discussion proves that this is not the case. Various theories of relativity can exist. It is possible to construct consistent nonlinear field theories, having the following properties: speed of information transfer is finite one but the Lorentz symmetry is broken. The next important conclusion is the following one: particle-like superluminal soliton solutions (tachyons) are possible in such theories. Their existence does not break the causality principle and does not result in paradoxes. All relativistic effects in nonlinear field theories are similar to usual relativistic effects of the special theory of relativity (for example, width of a soliton decreases with growth of its velocity) but the mathematical description of these effects differs from conventional Lorentz one. One more important conclusion: there are nonlinear field theories where breaking Lorentz symmetry is essential one if soliton velocity is close to velocity of light. If soliton velocity tends to zero then its properties asymptotically approach properties of usual particles described by the special theory of relativity.

The hypothesis that space-time has discrete (quantized) structure has been discussed for a long time [19]. The basic lack of the theories, based on this hypothesis, is the contradiction between discrete structure of space-time and Lorentz symmetry. We have shown that this contradiction is absent in the theory of solitons in condensed matter. The theory of solitons in solids is a promising model for construction of field theories in discrete space-time.


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