MATHEMATICAL DERIVATION OF THE FARADAY INDUCTION LAW AND EXPLANATION OF ITS LORENTZ NON-INVARIANCE

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The present paper derives the Faraday induction law mathematically and discusses a problem of its Lorentz non-invariance.

It is known that the Faraday induction law

$$\varepsilon = -\frac{d}{dt}\int_S \vec{B}d\vec{S} \tag{1}$$

is valid for both fixed and variable closed circuit Γ, restricting the area *S*. For fixed Γ,

$$\varepsilon = \oint_\Gamma \vec{E}(\vec{r},t)d\vec{l} \ , \tag{2}$$

for variable with time circuit ($\Gamma = \Gamma(t)$),

$$\varepsilon = \oint_{\Gamma(t)} \left(\vec{E}(\vec{r},t) + \vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)d\vec{l} \ , \tag{3}$$

where $\vec{r}$ is the radius-vector within the circuit Γ in the reference frame considered. Eqs. (2) and (3) follow from the definition of e.m.f.

$$\varepsilon = \oint_\Gamma \vec{f}d\vec{l} \ , \tag{4}$$

(*f* is the force acting on a unit charge) and the Lorentz force law.

Substituting Eq. (3) into Eq. (1), one gets:

$$\oint_{\Gamma(t)} \left(\vec{E}(\vec{r},t) + \vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)d\vec{l} = -\frac{d}{dt}\int_{S(t)} \vec{B}d\vec{S} \ . \tag{5}$$

Considering Eq. (4) in his classical text-book [1], Jackson mentions a surprising fact: this equation is not Lorentz-invariant for Γ, changing with time. At the same time, this very interesting fact was dispensed with attention of the researchers during a long time. In this contribution we will look more closely on the Faraday induction law in a light of its non-invariance.

First of all, let us express the Faraday induction law via the potentials of electromagnetic (EM) field:

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\varphi \ , \ \vec{B} = \nabla\times\vec{A} \ . \tag{6}$$

Substituting these equations into (5), we obtain

$$\oint_{\Gamma(t)} \left(-\frac{\partial\vec{A}}{\partial t} - \nabla\varphi + \vec{v}\times\left(\nabla\times\vec{A}\right)\right)d\vec{l} = -\frac{d}{dt}\int_S \left(\nabla\times\vec{A}\right)d\vec{S} \ . \tag{7}$$

Taking into account that:

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\nabla)\vec{A} \ , \tag{8}$$

$$\vec{v}\times\left(\nabla\times\vec{A}\right)=\nabla\left(\vec{v}\vec{A}\right)-\left(\vec{v}\nabla\right)\vec{A}$$

$$\oint_{\Gamma(t)}\nabla\varphi d\vec{l}=0, \qquad (9)$$

$$\oint_{\Gamma(t)}\nabla\left(\vec{v}\vec{A}\right)d\vec{l}=0, \qquad (10)$$

and applying the Stokes theorem to *rhs* of Eq. (7), we derive:

$$\oint_{\Gamma(t)}\frac{d\vec{A}}{dt}d\vec{l}=\frac{d}{dt}\oint_{\Gamma(t)}\vec{A}d\vec{l}, \qquad (11)$$

which, in general, looks mathematically incorrect for changing with time Γ.

For further analysis let us prove a theorem as follows:

- for any smooth vector field $\vec{A}(\vec{r},t)$ and any smooth closed line $\Gamma(t)$,

$$\frac{d}{dt}\oint_{\Gamma(t)}\vec{A}d\vec{l}=\oint_{\Gamma(t)}\left(\frac{d\vec{A}}{dt}+\vec{A}\times\left(\nabla\times\vec{v}\right)+\left(\vec{A}\nabla\right)\vec{v}\right)d\vec{l}, \qquad (12)$$

where $\vec{v}=\vec{v}\left(\vec{r},t\right)$ is the velocity of the point $\vec{r}$ within the line Γ.

Proof:

Let us write the integral $\oint_{\Gamma(t)}\vec{A}d\vec{l}$, proceeding from its definition:

$$\oint_{\Gamma(t)}\vec{A}d\vec{l}=\oint_{\Gamma(t)}\left(A_x\left(\vec{r},t\right)dx+A_y\left(\vec{r},t\right)dy+A_z\left(\vec{r},t\right)dz\right)=$$

$$=\lim_{n\to\infty}\left[\sum_{i=1}^{n}A_x\left(\vec{r}_i,t\right)\Delta x_i+\sum_{i=1}^{n}A_y\left(\vec{r}_i,t\right)\Delta y_i+\sum_{i=1}^{n}A_z\left(\vec{r}_i,t\right)\Delta z_i\right],$$

where $\Delta x_i=x_i-x_{i-1}$, $\Delta y_i=y_i-y_{i-1}$, $\Delta z_i=z_i-z_{i-1}$ are the components of elemental lines on the corresponding coordinate axes, and $\vec{r}_i$ is the coordinate of any point on the elemental line *i*. Then

$$\frac{d}{dt}\oint_{\Gamma(t)}\vec{A}d\vec{l}=\frac{d}{dt}\oint_{\Gamma(t)}\left(A_x\left(\vec{r},t\right)dx+A_y\left(\vec{r},t\right)dy+A_z\left(\vec{r},t\right)dz\right)=$$

$$=\lim_{n\to\infty}\left[\frac{d}{dt}\sum_{i=1}^{n}A_x\left(\vec{r}_i,t\right)\Delta x_i+\frac{d}{dt}\sum_{i=1}^{n}A_y\left(\vec{r}_i,t\right)\Delta y_i+\frac{d}{dt}\sum_{i=1}^{n}A_z\left(\vec{r}_i,t\right)\Delta z_i\right]=$$

$$=\lim_{n\to\infty}\left[\begin{array}{l}\sum_{i=1}^{n}\frac{dA_x}{dt}\left(\vec{r}_i,t\right)\Delta x_i+\sum_{i=1}^{n}\frac{dA_y}{dt}\left(\vec{r}_i,t\right)\Delta y_i+\sum_{i=1}^{n}\frac{dA_z}{dt}\left(\vec{r}_i,t\right)\Delta z_i+\sum_{i=1}^{n}A_x\left(\vec{r}_i,t\right)\left(\frac{dx_i}{dt}-\frac{dx_{i-1}}{dt}\right)+\\+\sum_{i=1}^{n}A_y\left(\vec{r}_i,t\right)\left(\frac{dy_i}{dt}-\frac{dy_{i-1}}{dt}\right)+\sum_{i=1}^{n}A_z\left(\vec{r}_i,t\right)\left(\frac{dz_i}{dt}-\frac{dz_{i-1}}{dt}\right)\end{array}\right]=$$

$$=\lim_{n\to\infty}\left[\begin{array}{l}\sum_{i=1}^{n}\frac{dA_x}{dt}\left(\vec{r}_i,t\right)\Delta x_i+\sum_{i=1}^{n}\frac{dA_y}{dt}\left(\vec{r}_i,t\right)\Delta y_i+\sum_{i=1}^{n}\frac{dA_z}{dt}\left(\vec{r}_i,t\right)\Delta z_i+\sum_{i=1}^{n}A_x\left(\vec{r}_i,t\right)\left(\left(v_x\right)_i-\left(v_x\right)_{i-1}\right)+\\+\sum_{i=1}^{n}A_y\left(\vec{r}_i,t\right)\left(\left(v_y\right)_i-\left(v_y\right)_{i-1}\right)+\sum_{i=1}^{n}A_z\left(\vec{r}_i,t\right)\left(\left(v_z\right)_i-\left(v_z\right)_{i-1}\right)\end{array}\right]=$$

$$=\oint_{\Gamma(t)}\frac{d\vec{A}}{dt}\left(\vec{r},t\right)d\vec{l}+\lim_{n\to\infty}\left[\sum_{i=1}^{n}A_x\left(\vec{r}_i,t\right)\Delta\left(v_x\right)_i+\sum_{i=1}^{n}A_y\left(\vec{r}_i,t\right)\Delta\left(v_y\right)_i+\sum_{i=1}^{n}A_z\left(\vec{r}_i,t\right)\Delta\left(v_z\right)_i\right].$$

Further, taking into account that for any fixed time moment the velocities $\vec{v}_i$ can be considered as the smooth functions of $\vec{r}_i$ within the closed line $\Gamma(t)$, we can write

$$\Delta(v_x)_i = \frac{\partial v_x}{\partial x}\Delta x_i + \frac{\partial v_x}{\partial y}\Delta y_i + \frac{\partial v_x}{\partial z}\Delta z_i ,$$

$$\Delta(v_y)_i = \frac{\partial v_y}{\partial x}\Delta x_i + \frac{\partial v_y}{\partial y}\Delta y_i + \frac{\partial v_y}{\partial z}\Delta z_i$$

$$\Delta(v_z)_i = \frac{\partial v_z}{\partial x}\Delta x_i + \frac{\partial v_z}{\partial y}\Delta y_i + \frac{\partial v_z}{\partial z}\Delta z_i .$$

From there we obtain

$$\frac{d}{dt}\oint_{\Gamma(t)}\vec{A}d\vec{l} = \oint_{\Gamma(t)}\frac{d\vec{A}}{dt}(\vec{r},t)d\vec{l} + \oint_{\Gamma(t)}A_x\left(\frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz\right) + \oint_{\Gamma(t)}A_y\left(\frac{\partial v_y}{\partial x}dx + \frac{\partial v_y}{\partial y}dy + \frac{\partial v_y}{\partial z}dz\right) +$$

$$+ \oint_{\Gamma(t)}A_z\left(\frac{\partial v_z}{\partial x}dx + \frac{\partial v_z}{\partial y}dy + \frac{\partial v_z}{\partial z}dz\right) =$$

$$= \oint_{\Gamma(t)}\frac{d\vec{A}}{dt}(\vec{r},t)d\vec{l} + \oint_{\Gamma(t)}\left(A_x\frac{\partial v_x}{\partial x} + A_y\frac{\partial v_y}{\partial x} + A_z\frac{\partial v_z}{\partial x}\right)dx + \left(A_x\frac{\partial v_x}{\partial y} + A_y\frac{\partial v_y}{\partial y} + A_z\frac{\partial v_z}{\partial y}\right)dy +$$

$$+ \oint_{\Gamma(t)}\left(A_x\frac{\partial v_x}{\partial z} + A_y\frac{\partial v_y}{\partial z} + A_z\frac{\partial v_z}{\partial z}\right)dz =$$

$$= \oint_{\Gamma(t)}\frac{d\vec{A}}{dt}(\vec{r},t)d\vec{l} + \oint_{\Gamma(t)}\nabla_v\left(\vec{v}\vec{A}\right)d\vec{l} , \qquad (13)$$

where the operator ∇_v acts only on $\vec{v}$. It can be expressed via the conventional operator ∇ as

$$\nabla_v\left(\vec{v}\vec{A}\right) = \vec{A}\times(\nabla\times\vec{v}) + \left(\vec{A}\nabla\right)\vec{v} . \qquad (14)$$

Substituting its value into Eq. (13) one gets Eq. (12), that proves the theorem.

Using the obtained theorem, we transform the expression (11) into

$$\oint_{\Gamma(t)}\frac{d\vec{A}}{dt}d\vec{l} = \frac{d}{dt}\oint_{\Gamma(t)}\left[\frac{d\vec{A}}{dt} + \vec{A}\times(\nabla\times\vec{v}) + \left(\vec{A}\nabla\right)\vec{v}\right]d\vec{l} , \qquad (15)$$

and this equation is obviously incorrect. On the other hand, it has been derived from the Faraday induction law, which has a perfect experimental confirmation! It means that we have made an error under derivation of Eq. (15) and Eq. (11) as well.

In order to reveal such an error, we again put the attention on the fact that the proved theorem has to operate with the vector field $\vec{v}(\vec{r},t)$, defined on a closed line Γ. Hence, under derivation of the *lhs* of Eqs. (11) and (15) we have also to take into account that the parameter $\vec{v}$ is no longer a conventional velocity, but rather the field of velocities, defined on Γ. Then, in order to obtain an expression for e.m.f. in a correct way, we proceed from the definition (4), as well as from the motional equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{v}}\right)_{\vec{r}} = \left(\frac{\partial L}{\partial\vec{r}}\right)_{\vec{v}} , \qquad (16)$$

where L is the Lagrange function for a charged particle in electromagnetic field, and the corresponding subscripts indicate the non-variable parameter. Using the conventional expression for L (see, *e.g.* [4]), we obtain for the force per unit charge:

$$\vec{f} = -\frac{d\vec{A}}{dt} + \nabla_A\left(\vec{v}\vec{A}\right). \tag{17}$$

Here it is essential, that operator of gradient (∇_A) acts only on $\vec{A}$, but not on the vector field $\vec{v}$. Hence, the circular integral of this operator in not, generally, vanishing. It means that Eq.(10) is no longer correct, when we deal with the field of velocities. Comparing Eq. (17) and Eq. (4), we obtain

$$\varepsilon = \oint\limits_{\Gamma(t)} \left(-\frac{d\vec{A}}{dt} + \nabla_A\left(\vec{v}\vec{A}\right) \right) d\vec{l}\ . \tag{18}$$

It means that Eq. (11) transforms to

$$\oint\limits_{\Gamma(t)} \left(-\frac{d\vec{A}}{dt} + \nabla_A\left(\vec{v}\vec{A}\right) \right) d\vec{l} = -\oint\limits_{\Gamma(t)} \left[\frac{d\vec{A}}{dt} + \vec{A}\times(\nabla\times\vec{v}) + \left(\vec{A}\nabla\right)\vec{v} \right] d\vec{l}\ ,$$

or

$$\oint\limits_{\Gamma(t)} \left(-\frac{d\vec{A}}{dt} + \nabla_A\left(\vec{v}\vec{A}\right) \right) d\vec{l} = \oint\limits_{\Gamma(t)} \left[-\frac{d\vec{A}}{dt} - \nabla_v\left(\vec{v}\vec{A}\right) \right] d\vec{l}\ . \tag{19}$$

Noting that $\nabla_A\left(\vec{v}\vec{A}\right) + \nabla_v\left(\vec{v}\vec{A}\right) = \nabla\left(\vec{v}\vec{A}\right)$, and for the total gradient

$$\oint \nabla\left(\vec{v}\vec{A}\right) d\vec{l} = 0\,,$$

we prove the equality (19).

Simultaneously it means that now we are able to derive the Faraday induction law mathematically. Indeed, going from the vector potential to the electric and magnetic fields in Eq. (19) with application of the equality

$$\nabla_A\left(\vec{v}\vec{A}\right) = \vec{v}\times\left(\nabla\times\vec{A}\right) + \left(\vec{v}\nabla\right)\vec{A},$$

we have

$$\oint\limits_{\Gamma(t)} \left[-\frac{d\vec{A}(\vec{r},t)}{dt} + \vec{v}(\vec{r},t)\times\left(\nabla\times\vec{A}(\vec{r},t)\right) + \left(\vec{v}(\vec{r},t)\nabla\right)\vec{A}(\vec{r},t) \right] d\vec{l} = -\frac{d}{dt}\oint\limits_{\Gamma(t)} \vec{A}(\vec{r},t)\,d\vec{l}\ ,$$

or

$$\oint\limits_{\Gamma(t)} \left[-\frac{\partial\vec{A}(\vec{r},t)}{\partial t} + \vec{v}(\vec{r},t)\times\left(\nabla\times\vec{A}(\vec{r},t)\right) \right] d\vec{l} = -\frac{d}{dt}\int\limits_{S} \left[\nabla\times\left(\vec{A}(\vec{r},t)\right)\right] d\vec{S}\ ,$$

or

$$\oint\limits_{\Gamma(t)} \left[\vec{E} + \vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)\right] d\vec{l} = -\frac{d}{dt}\int\limits_{S(t)} \vec{B}(\vec{r},t)\,d\vec{S}\ , \tag{20}$$

or

$$\varepsilon = -\frac{d}{dt}\int\limits_{S(t)} \vec{B}(\vec{r},t)\,d\vec{S}\,.$$

This result admits to say that Eq. (19) represents the Faraday induction law, expressed though the vector potential.

Thus, introducing of the vector field $\vec{v}(\vec{r},t)$ allows one to understand a mathematical structure of the Faraday induction law. The appearance of this field in *lhs* of Eq. (20) instead of conventional velocity parameter makes this equation to be Lorentz non-invariant. The reason of its non-invariance is an impossibility, in general, to turn to zero the field $\vec{v}(\vec{r},t)$ in all spatial points under the Lorentz transformations, excepting a trivial case $\vec{v}(\vec{r},t)$=const. On the other hand, the magnetic field, in principle, can be turned to zero in all spatial points. Then the Lorentz non-invariance of the Faraday law is especially clearly seen in a stationary case, when $\vec{B}(\vec{r},t)=\vec{B}(\vec{r})$, and the induced electric field is equal to zero:

$$\varepsilon = \oint_{\Gamma(t)} \left[\vec{v}(\vec{r},t)\times \vec{B}(\vec{r})\right] d\vec{l} \ . \tag{21}$$

If the Column electric field in the inertial frame considered is orthogonal to the magnetic field, and its value *E*>*Bc*, that one can always find an inertial frame, where *B*=0, and ε=0. On the other hand, in another inertial frame, in general, *B*≠0, and also ε≠0. This result contradicts to the relativistic transformation of e.m.f. [3]

$$\varepsilon' = \varepsilon\sqrt{1-\frac{u^2}{c^2}}\ , \tag{22}$$

where *u* is a relative velocity between two inertial reference frames (here ε belongs to a resting (laboratory) frame). This transformation satisfies to the causal requirements: 1 - the e.m.f. simultaneously vanishes in all inertial frames; 2- if ε≠0, that its sign is the same for all observers. However, one sees that the e.m.f. in Eq. (21) falls outside these requirements.

The aforesaid can be demonstrated by a particular example (Fig. 1).

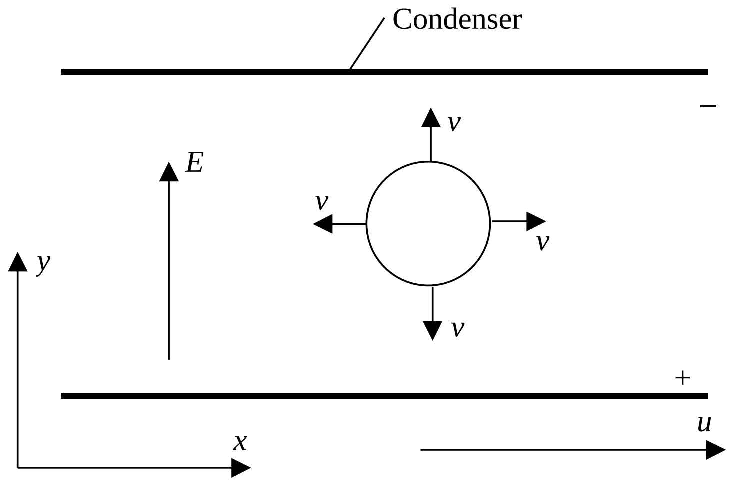

Fig. 1. Expanded circumference is inside the parallel plate charged capacitor.

An expanded circumference with the radius *R(t)=R(0)+vt* is placed inside the parallel plate capacitor. In the rest frame of capacitor, only the constant electric field *E* exists along the axis *y*, and there is no e.m.f. in the loop of expanded circumference. However, for an observer in the frame K, wherein the capacitor and circumference move at the constant velocity *u* along the axis *x*, the e.m.f. is not equal to zero. Indeed, in the frame K

$$B_z = \frac{uE}{c^2\sqrt{1-u^2/c^2}}\ ,$$

and

$$\varepsilon = \oint_{\Gamma} vB_z dl = 2\pi R(t)\frac{uvE}{c^2}\sqrt{1-u^2/c^2} \neq 0.$$

Another reason, which causes the Lorentz non-invariance of the Faraday induction law, is a possible discontinuity of the vector field $\vec{v}(\vec{r})$ upon Γ (i.e., Γ is not a smooth closed line). This case is realized under relative slip of the parts of the loop Γ (see, e.g., Fig. 2). In Fig. 2 the vector field in a laboratory frame is reduced to a single velocity *v* of a motion of crosspiece AB along the axis *x*. At the same time, the points A and B represent the points of discontinuity of the velocity, defined upon the closed line A-B-C-D, and the known identities of vector analysis becomes invalid under integration though this closed line. (In particular, a circular integral of the total gradient $\nabla(\vec{v}\vec{A})$ is not vanished). It leads to the Lorentz non-invariance of the Faraday induction law, too. For instance, in the rest frame of charged flat capacitor FT the e.m.f. in the loop A-B-C-D is equal to zero, while in the inertial frame K, wherein the capacitor and loop move at the constant velocity *u* along the axis *x*, the e.m.f. is defined as [4]

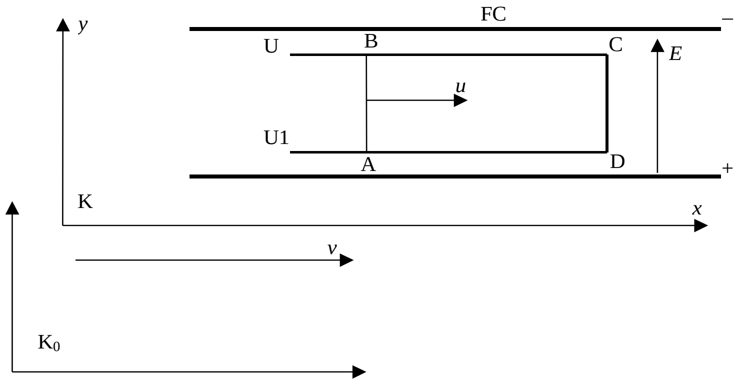

Fig. 2. The rectangular closed line A-B-C-D with moving segment AB inside the charged flat condenser FC. The e.m.f. in the circuit is equal to zero in the rest frame of condenser K, and it is not equal to zero in the frame K_0, where the frame K moves at the constant velocity *v* along the axis *x*. It means a violation of the Lorentz invariance.

$$\varepsilon = \frac{uvEl\sqrt{1-u^2/c^2}}{c^2\left(1+uv/c^2\right)},$$

where *l* is the length of AB. We again see a violation of the Lorentz-invariance.

At the same time, one needs to stress that the results obtained are correct only under integration over the closed mathematical lines in space. That is why we now investigated only a mathematical structure of the Faraday induction law, but not its physics. In particular, under substitution of the expanded circumference (Fig. 1) by a conductive closed circuit, the electrons in a conductor are re-arranged by such a way, in order to give a vanishing electric field inside the conductor. Then for the rest frame of capacitor we have to write *E*,*B*=0 inside the conductor. Due to a homogeneity of field transformations, the same equality holds true in any other inertial frame. Hence, the induced e.m.f. is equal to zero for any inertial observer in accordance with the Einstein relativity principle. The same remark is also suitable for the problem in Fig. 2.

Conclusion

Thus, we have shown that the introducing of the field of velocities, being defined on a closed mathematical line, allows one to derive the Faraday induction law mathematically. The appearance of field of velocities makes the Faraday induction law to be not Lorentz-invariant. At the same time, this result is always true though integration over the mathematical closed lines in space. In real conductors one needs to take into account their polarization, which recovers the Einstein relativity principle for the considered above physical problems. The physical problems, where polarization of conductors does not contribute the e.m.f., will be considered in a separate paper.

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