Losslessness in Nonlinear Kirchhoff Circuits and in Relativity Theory

Alfred Fettweis

Department of Electrical Engineering and Information Science/Communications Engineering,
Ruhr-Universitaet Bochum,
D-44780 Bochum, Germany,
fettweis@nt.ruhr-uni-bochum.de.

Abstract

Kirchhoff circuits are of importance not only for studying electrical phenomena but are ideally suited to model a broad range of physical systems for purposes where conservation of power and energy and related concepts such as passivity and losslessness are essential. They consist of interconnections of a variety of elements, which nowhere have to be linear and constant. If an element such as an inductance is nonlinear and/or explicitly time dependent and is to be characterized as being passive or, more specifically, lossless, its defining relation must have a specific form, but the classical relation for a relativistic mass is not of this type. It is shown that, preserving classical relativistic kinematics, requiring relativistic dynamics to approach the Newtonian one for appropriate limits, and putting prime emphasis on work done, thus on energy rather than momentum, one is naturally led to an expression for force in terms of mass and velocity whose form is in full agreement with that referred to for a nonlinear inductance. This alternative way of modifying Newton's second law requires Newton's third law to be also modified. These two modifications combined produce the same conservation of momentum and the same dynamics of particles in fields as classical relativity. The expression for kinetic energy, however, is different. Logically consistent derivations are presented, and a theoretical and an experimental result are pointed out that tend to offer some support to the alternative theory, or at least do not contradict it, as implausible as that theory may a priori appear to be. The paper complements and updates earlier results on the subject and improves the presentation.

1 Introduction

This paper is dedicated to the memory of the great physicist and chemist Ilya Progogine. His oustanding contributions have touched my own activities in at last two important ways: On the one hand he is particularly known for his work on non-equilibrium thermodynamics, thus an area in which the theory of an entirely different discipline, i.e. that of Kirchhoff circuits [1, 2], can play a helpful role [3]. On the other hand, he is one of the outstanding Russian émigrés who came to Belgium as boys in the aftermath of the Bolshevikh revolution and who have contributed so much to the scientific life of that country. Another one of those was Vitold

Belevitch [4], who has been the dominant figure in the theory of Kirchhoff circuits during the second half of the twentieth century, certainly in Europe and, in the opinion of many, also worldwide, and who is the grand master to whom this author owes the privilege of having been introduced into the beauty, generality, and rigor of that theory. The present paper shows how the theory of Kirchhoff circuits can throw some new light even on certain aspects of relativity theory.

Kirchhoff circuits are of importance not only in the context of modelling electric phenomena but also for a variety of other reasons. As an example, they can serve as reference circuits for applying wave-digital concepts, in particular to filtering [5] and numerical integration of ordinary [6] and, especially, partial differential equations describing physical systems [7 - 9]. While for filtering one is usually interested in linear circuits, numerical integration largely addresses nonlinear phenomena.

Kirchhoff circuits are composed of elements that are interconnected by merging element terminals into nodes. The interconnections are described by the Kirchhoff voltage and current laws. A crucial consequence of this is that conservation of power and thus energy is strictly guaranteed by the validity of the Kirchhoff laws. Hence, properties such as passivity and losslessness (which is a special case of passivity) are in turn strictly ensured if the corresponding properties hold for all the elements. In order to guarantee in a simple way that an element such as a nonlinear inductance is lossless, however, it should be represented in a somewhat different way than what is usually done [7 - 9], and this alternative representation turns out to be of rather fundamental physical importance.

Mass is, in a sense, also a lossless (conservative) element. In Newtonian mechanics this element is linear, but in relativistic mechanics it is nonlinear. Surprisingly, however, the relationship between force and velocity in classical relativity theory [10 - 19] does not have the general form one might expect in view of what has just been said about a nonlinear inductance.

In previous papers [20, 21] it has been shown that by making certain assumptions that at least are not a priori unreasonable one is naturally led to a relativistic expression between force, mass, and velocity that differs somewhat from the classical one but has a form that is in full agreement with the general form just referred to. This discrepancy is essentially due to the fact that in classical relativity, momentum is the quantity placed at the beginning of the derivations for establishing the laws of dynamics, while the alternative approach puts prime emphasis on energy and work done. There is no change concerning the use of the Lorentz transformation and thus the validity of relativistic kinematics, which is indeed known to be the result of intellectually highly satisfying and strict logical deductions. We thus maintain the second Einstein postulate, i.e. the one concerning the universality of the speed of light. Similarly, we uphold also the first one, i.e. the one concerning the universality of the fundamental laws of nature. In a sense, we do however complement the two Einstein postulates by what we will refer to as the *principle of Newtonian limit*.

The alternative modification of Newton's 2nd law (Section 4) requires a corresponding modification of his 3rd law (Section 5), although the latter remains untouched in classical relativity. The two changes combined imply that conservation of momentum holds as in classical relativity (Section 5) and that the dynamics of particles in fields (e.g. in particle accelerators) is also the same (Section 6). An expression, obtainable by standard approaches, for the energy of an electrostatic field observed in a moving reference frame (Section 7) agrees with the alternative theory. The Bertozzi experiment (Section 8) does not, to say the least, disprove it either, although that experiment involves kinetic energy. The expression for that quantity is indeed different in the two cases so that there have to exist ways of differentiating.

The present paper pursues several prime purposes. Firstly, the main result, i.e. the derivation of the alternative relation between force, velocity, and mass, is carried out in a somewhat different way than previously done, thus offering an even firmer basis for justifying that expression. Secondly, the Bertozzi experiment is analyzed more precisely than has been done in [21] and is shown to actually have a tendency of favoring the alternative approach, although the accuracy of the experiment is not high enough to allow us to draw definite conclusions. Thirdly, several inaccuracies in [21] are corrected, but we keep aiming at the same generality. Fourthly, the present paper is not only an improved but also a substantially expanded version of [22] and should thus be more easily accessible. Of course, as in [21] but contrary to what had been done in [20], we do not impose any restrictions on the integration constant occurring in relation with the energy expression. All previous results that are essential for understanding the paper are included for ease of presentation and thus for the benefit of the reader.

Clearly, the entire issue remains puzzling. It is hoped that the results presented hereafter will stimulate further experimental and theoretical investigations that bring about the needed clarification.

2 Passivity and losslessness in nonlinear Kirchhoff circuits

A nonlinear inductance that depends exclusively on its current, i, can be described by either one of the equations

$$u = \mathrm{D}(L_g i) \quad \text{or} \quad u = L_l \mathrm{D} i \tag{2.1}$$

where u is the voltage across the inductance and

$$\mathrm{D} = \mathrm{d}/\mathrm{d}t, \tag{2.2}$$

$L_g = L_g(i) = \phi/i$ being the *global inductance* and $L_l = L_l(i) = \mathrm{d}\phi/\mathrm{d}i$ the *local inductance*, and ϕ the magnetic flux. It can be shown that $L_g(i) \geq 0 \quad \forall i$ is necessary but not sufficient for guaranteeing passivity (in fact losslessnes) while $L_l(i) \geq 0 \quad \forall i$ is sufficient but not necessary. In order to obtain a satisfactory characterization Meerkötter [23, 24] had suggested to represent a nonlinear inductance by means of a nonlinear ideal transformer terminated in a linear inductance, but a simpler, equivalent way is to write (2.1) in the form [7 - 9]

$$u = \sqrt{L}\mathrm{D}(\sqrt{L}i) = \frac{1}{2}(\mathrm{D}(Li) + L\mathrm{D}i). \tag{2.3}$$

The two expressions in (2.3) correspond, in a sense, to the the geometric and the arithmetic mean of the definitions (2.1). For the power absorbed and the stored energy, W_L, one finds from (2.3),

$$ui = \mathrm{D}W_L, \qquad W_L = \frac{1}{2}Li^2, \tag{2.4}$$

and for the *inductance*, L, the inequality $L(i) \geq 0$ is now necessary and sufficient for passivity (in fact, losslessness).

Writing (2.3) in the form $x = \mathrm{D}y, \quad x = u/\sqrt{L}, \quad y = i\sqrt{L}$, approximating D by means of the trapezoidal rule with step size T, and designating the discretized time variable by t_n, the result can be written in the form

$$b(t_n) = -a(t_n - T) \tag{2.5}$$

where the so-called *waves* (wave quantities), a and b, are given by

$$a = \frac{u + iR}{2\sqrt{R}}, \quad b = \frac{u - iR}{2\sqrt{R}}, \quad R = \frac{2L}{T}. \tag{2.6}$$

We obviously also have

$$ui = a^2 - b^2. \tag{2.7}$$

All this holds more generally if L is a function not only of i but of any of the dependent variables in the circuit and/or of the independent variable, say t. It holds even for multidimensional Kirchhoff circuits [7 - 9], in which case D is a partial differential operator and t has to be replaced by a vector of independent variables.

The simplicity of expressions such as (2.5) and (2.7) and their combination with passivity and losslessness aspects are the essential reasons for the advantageous properties that can be obtained by making use of wave-digital principles for filtering [5] and numerical integration of ordinary and partial differential equations [6 - 9]. The inductance representation (2.3) can be extended to coupled inductances, thus to matrix inductances $\mathbf{L}$, that matrix being symmetric and non-negative definite. With $\mathbf{u}$ and $\mathbf{i}$ being vector extensions of u and i one can then write,

$$\mathbf{u} = \frac{1}{2}\left(\mathbf{L}\mathrm{D}\mathbf{i} + \mathrm{D}\left(\mathbf{L}\mathbf{i}\right)\right) \qquad \text{or} \qquad \mathbf{u} = \mathbf{L}^{\mathrm{T}/2}\mathrm{D}\left(\mathbf{L}^{1/2}\mathbf{i}\right) \tag{2.8}$$

where the matrix $\mathbf{L}^{1/2}$ is such that $\mathbf{L}^{\mathrm{T}/2}\mathbf{L}^{1/2} = \mathbf{L}$, the superscript T designating transposition and $\mathbf{L}^{\mathrm{T}/2} = (\mathbf{L}^{1/2})^{\mathrm{T}}$. The two expressions in (2.8) are equivalent if and only if $\mathbf{L}^{\mathrm{T}/2}\mathrm{D}(\mathbf{L}^{1/2}) = (\mathrm{D}\mathbf{L}^{\mathrm{T}/2})\mathbf{L}^{1/2}$. The general conditons under which the latter equality holds are not known [25], but it is sufficient that $\mathbf{L}^{1/2}$ is a diagonal matrix times a constant matrix (cf. the situation encountered later in relation with (3.13)).

The wide importance energy, wave, and scattering concepts have in physics suggests that relations such as (2.3) and (2.4) have some fundamental physical importance. This observation, however, is not honored by the classical theory of special relativity.

3 Relativistic mass

For a particle of *rest mass* m_0, let

$$\mathbf{f} = (f_x, f_y, f_z)^{\mathrm{T}}, \quad \mathbf{p} = (p_x, p_y, p_z)^{\mathrm{T}}, \quad \mathbf{v} = (v_x, v_y, v_z)^{\mathrm{T}} \tag{3.1}$$

be the force acting upon it, its momentum, and its velocity, respectively. Define v, β, and α by

$$\mathbf{v}^{\mathrm{T}}\mathbf{v} = v^2, \quad \beta = v/c, \quad \alpha = \sqrt{1 - \beta^2} \tag{3.2}$$

where c is the speed of light. According to the basic principles adopted in classical relativity theory the following holds (cf. (2.2)),

$$\mathbf{f} = \mathrm{D}\mathbf{p}, \quad \mathbf{p} = m_g\mathbf{v}, \quad m_g = m_0/\alpha \tag{3.3}$$

where we have written m_g instead of m in order to conform with the notation in (2.1). From (3.3) one finds for the power delivered to the particle, as is well known,

$$\mathbf{v}^{\mathrm{T}}\mathbf{f} = \mathrm{D}\left(m_g c^2\right), \tag{3.4}$$

and this by making use of the identity

$$c^2 \mathrm{D} \frac{1}{\alpha} = \mathbf{v}^{\mathrm{T}} \mathrm{D} \left(\frac{1}{\alpha} \mathbf{v} \right). \tag{3.5}$$

The quantity $m_g c^2$ is then interpreted as total energy while $(m_g - m_0)\, c^2$ is the kinetic energy.

Clearly, m_g as defined by (3.3) does indeed correspond to L_g in (2.1), not to L in (2.3). This is somewhat surprising and, as explained in [20 - 22], the reason for examining whether an expression for the *force* such as

$$\mathbf{f} = \sqrt{m} \mathrm{D} \left(\sqrt{m} \mathbf{v} \right) = \frac{1}{2} \left(\mathrm{D}(m\mathbf{v}) + m \mathrm{D} \mathbf{v} \right) \tag{3.6}$$

where the *mass* m is some function of v, say $m = m(\beta)$, could be of interest. In the affirmative, (3.4) would be replaced by

$$\mathbf{v}^{\mathrm{T}} \mathbf{f} = \mathrm{D} E_k = \mathrm{D} E \tag{3.7}$$

where

$$E = E_i + E_k, \quad E_k = mv^2/2, \quad E_i = \mathrm{const.} \tag{3.8}$$

Since $E_k = 0$ for $v = 0$, E_k would be the *kinetic energy*, thus E_i the *rest energy* (internal energy) and E the *total energy*. One could then assume m, not m_g, to be the quantity of prime interest. In particular, defining a rest mass $m_0 = m(0)$ one would want to preserve essential properties of classical relativity, although for m instead of m_g, and require that $m - m_0$ increases with E_k, preferably in a linear fashion, i.e., according to

$$E_k = K(m - m_0), \quad K = \mathrm{const.},$$

and that $m = \infty$ for $v = c$. It then follows from the above expression for E_k (c.f. (3.8)) that $K = c^2/2$, that m must thus be of the form

$$m = \frac{m_0}{1 - \beta^2} = \frac{m_0}{\alpha^2}, \tag{3.9}$$

and that therefore $E_k = \frac{1}{2} \left(mc^2 - m_0 c^2 \right)$.

In view of $E_k = E - E_i$, that last result had suggested, in the earlier paper [20], to identify E with $mc^2/2$ and thus E_{i} with $m_0 c^2/2$. This, however, leads to difficulties, while nothing contradicts the above equations if one does not impose any requirement on E_{i}, i.e., on the integration constant implied by (3.7). E and E_k are then still given by (3.8), thus E_k by

$$E_k = \frac{1}{2} mv^2 = E_0 \frac{\beta^2}{1 - \beta^2} = E_0 \frac{\beta^2}{\alpha^2}, \;\; E_0 = \frac{1}{2} m_0 c^2, \tag{3.10}$$

with m as defined in (3.9). The case discussed in [20] corresponds to choosing $E_{\mathrm{i}} = E_0$. Adopting the present point of view, however, amounts to saying that a particle is, in general, characterized by two constants, E_{i} and E_0. For $E_{\mathrm{i}} = 2E_0$, the rest energy is $E_{\mathrm{i}} = m_0 c^2$, as in the classical case. In general, however, no simple relation has to exist between E_{i} and E_0.

An attractive observation follows directly from (3.10). Using $v^2 = v_x^2 + v_y^2 + v_z^2$ the expression for E_k can indeed be decomposed into a sum of kinetic *energy components* as in Newtonian dynamics according to

$$E_k = E_{kx} + E_{ky} + E_{kz}$$

$$E_{kx} = \frac{1}{2}mv_x^2, \qquad E_{ky} = \frac{1}{2}mv_y^2, \qquad E_{kz} = \frac{1}{2}mv_z^2.$$

It is true that the triple $\mathbf{E}_k = (E_{kx}, E_{ky}, E_{kz})^{\mathrm{T}}$ does not form a vector in the strict physical sense (it transforms in a more complicated way than coordinates do, contrary to the momentum), but this should not be of any concern. Nevertheless this observation may partly explain why classical relativity gives preference to momentum for establishing the foundations of dynamics theory. In fact, the emphasis placed on energy components (i.e., not only the three kinetic ones mentioned above) is also of fundamental importance for the wave-digital method of numerically integrating partial differential equations [7 - 9].

By means of $\mathbf{f}$ one can define the quadruple

$$\begin{pmatrix} \mathbf{f} \\ \frac{1}{c}\ \mathbf{v}^{\mathrm{T}}\mathbf{f} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \frac{1}{c}\ \mathrm{D}E_k \end{pmatrix}, \tag{3.11}$$

for which in (3.11) a second, equivalent expression in terms of E_k (cf. (3.10)) is given and which can directly be compared to well-known results in relativity theory [10, 13 - 16, 19]. It can then easily be verified that this quadruple is identical to the four-vector (world vector, Minkowski force) originally introduced by Minkowski [27] and thus has a Lorentz invariant form. One immediate conclusion of this is that a Lorentz transformation (cf. Section 4) changes only that component of $\mathbf{f}$ that is oriented in the direction of the relative movement of the two reference frames with respect to each other, i. e., the components of $\mathbf{f}$ perpendicular to that direction remain unchanged, contrary to what holds in classical relativity theory.

Like for inductances (Section 2) one can even go a step further and consider a particle-like object characterized, in addition to E_i, by a constant nonnegative definite symmetric matrix $\mathbf{m}_0$ instead of simply a scalar mass m_0. The scalar mass m has then to be replaced by the *matrix mass* $\mathbf{m}$ given by (cf. (3.9)),

$$\mathbf{m} = \frac{1}{\alpha^2}\mathbf{m}_0. \tag{3.12}$$

Expression (3.6) for $\mathbf{f}$ becomes

$$\mathbf{f} = \mathbf{m}_0\frac{1}{\alpha}\mathrm{D}\left(\frac{1}{\alpha}\mathbf{v}\right) = \frac{1}{\alpha}\mathrm{D}\left(\mathbf{m}_0\frac{1}{\alpha}\mathbf{v}\right)$$

$$= \frac{1}{2}\left(\mathrm{D}(\mathbf{m}\mathbf{v}) + \mathbf{m}\mathrm{D}\mathbf{v}\right) = \frac{1}{\alpha}\mathbf{m}_0^{\mathrm{T}/2}\mathrm{D}\left(\frac{1}{\alpha}\mathbf{m}_0^{1/2}\mathbf{v}\right), \tag{3.13}$$

the adopted notation corresponding to that defined in relation with (2.8).

For the power delivered in the matrix case, (3.7) remains unchanged, but in (3.8) and (3.10) the expression for E_k has to be replaced by

$$E_k = \frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{m}\mathbf{v} = \frac{1}{2\alpha^2}\mathbf{v}^{\mathrm{T}}\mathbf{m_0}\mathbf{v}. \tag{3.14}$$

The matrix case reduces to the scalar one if $\mathbf{m}_0 = \mathbf{1}m_0$ where $\mathbf{1}$ is the unit matrix. An irreducible example for the matrix case will be mentioned in Section 7.

4 Direct derivation of the alternative way of modifying Newton's second law

4.1 General relations

In view of the surprising results of Section 3 it appears opportune to examine whether there exists a way of deriving them, and possibly further related ones, by some direct approach. Clearly, since energy played an essential role in Section 3, such a direct approach would have to be centered on energy and work done rather than on momentum, as in classical relativity.

Consider thus two reference frames, S and S', with S' moving with constant velocity v_0, say, in the x-direction of S. For S we use unprimed notation as in Section 3 (coordinates, velocities, forces, differential operators etc.) and for S' the corresponding primed notation. Between S and S' we thus have the Lorentz transformation [10 -19]

$$x' = \frac{1}{\alpha_0}\left(x - v_0 t\right), \quad y' = y, \quad z' = z, \qquad t' = \frac{1}{\alpha_0}\left(t - \beta_0 \frac{x}{c}\right) \tag{4.1}$$

where

$$\beta_0 = v_0/c, \quad \alpha_0 = \sqrt{1-\beta_0^2}. \tag{4.2}$$

We consider a particle, P, moving as discussed in Section 3. For P it is appropriate to introduce vectors of position coordinates

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(t) = (x, y, z)^{\mathrm{T}}, \\ \mathbf{r}' &= \mathbf{r}'(t') = (x', y', z')^{\mathrm{T}}, \end{aligned} \tag{4.3}$$

while t' can be uniquely expressed in terms of t, say $t' = t'(t)$, and vice versa. While (3.1) and (3.2) refer to S, we have, with respect to S',

$$\mathbf{f}' = \left(f'_x, f'_y, f'_z\right)^{\mathrm{T}}, \quad \mathbf{v}' = \left(v'_x, v'_y, v'_z\right)^{\mathrm{T}}, \tag{4.4}$$

$$\mathbf{v}'^{\mathrm{T}}\mathbf{v}' = v'^2, \quad \beta' = v'/c, \quad \alpha' = \sqrt{1-\beta'^2}, \tag{4.5}$$

where (cf. (2.2))

$$\mathbf{v} = \mathrm{D}\mathbf{r}, \quad \mathbf{v}' = \mathrm{D}'\mathbf{r}' \tag{4.6}$$

$$\mathrm{D} = \mathrm{d}/\mathrm{d}t, \quad \mathrm{D}' = \mathrm{d}/\mathrm{d}t'. \tag{4.7}$$

As can be verified by means of (4.1), we have, as known [10 - 19],

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\alpha'}{\alpha} = \frac{\alpha'_x}{\alpha_x}, \quad \alpha'\mathrm{D} = \alpha\mathrm{D}' \tag{4.8}$$

where, in addition to quantities already defined,

$$\alpha_x = \sqrt{1-\beta_x^2}, \quad \alpha'_x = \sqrt{1-{\beta'_x}^2}, \quad \beta_x = \frac{v_x}{c}, \quad \beta'_x = \frac{v'_x}{c}, \tag{4.9}$$

and furthermore,

$$v'_x = \frac{v_x - v_0}{1-\beta_x\beta_0}, \quad v'_y = \frac{\alpha'}{\alpha}v_y, \quad v'_z = \frac{\alpha'}{\alpha}v_z. \tag{4.10}$$

From these expressions, the following equalities can be derived:

$$\begin{aligned}\frac{1}{\alpha'^3}\mathrm{D}'v'_x &= \frac{1}{\alpha^3}\mathrm{D}v_x \\ &= \mathrm{D}\frac{v_x}{\alpha} + \frac{v_y^2+v_z^2}{\alpha^3c^2}\mathrm{D}v_x \\ &-\frac{v_x}{\alpha^3c^2}\left(v_y\mathrm{D}v_y + v_z\mathrm{D}v_z\right),\end{aligned} \tag{4.11}$$

$$\mathrm{D}'v'_y = \frac{\alpha'^2}{\alpha}\mathrm{D}\frac{v_y}{\alpha} + v'_y\frac{1}{\alpha}\mathrm{D}\alpha', \tag{4.12}$$

$$\mathrm{D}'v'_z = \frac{\alpha'^2}{\alpha}\mathrm{D}\frac{v_z}{\alpha} + v'_z\frac{1}{\alpha}\mathrm{D}\alpha', \tag{4.13}$$

$$\mathrm{D}\left(v_x/\alpha_x\right) = \mathrm{D}'\left(v'_x/\alpha'_x\right). \tag{4.14}$$

4.2 Newton's second law

Newton's first law is purely qualitative and always valid; it thus is irrelevant for our purpose. His second law concerns the expression of the force. For examining it we consider two reference frames S and S' as discussed in Section 4.1 and a particle P as so far. Let t_1 and t_2 be two time instants that are arbitrary close yet distinct, say $t_2 > t_1$. Quantities referring to t_1 and t_2 will be given subscripts 1 and 2, respectively, thus, e. g.,

$$\begin{array}{cccc} t'_1 = t'\left(t_1\right), & t'_2 = t'\left(t_2\right), & t_1 = t\left(t'_1\right), & t_2 = t\left(t'_2\right), \\ \mathbf{r}_1 = \mathbf{r}\left(t_1\right), & \mathbf{r}_2 = \mathbf{r}\left(t_2\right), & \mathbf{r}'_1 = \mathbf{r}'\left(t'_1\right), & \mathbf{r}'_2 = \mathbf{r}'\left(t'_2\right), \\ \mathbf{v}_1 = \mathbf{v}\left(t_1\right), & \mathbf{v}_2 = \mathbf{v}\left(t_2\right), & \mathbf{v}'_1 = \mathbf{v}'\left(t'_1\right), & \mathbf{v}'_2 = \mathbf{v}'\left(t'_2\right), \\ x_1 = x\left(t_1\right), & x_2 = x\left(t_2\right), & v_{x1} = v_x\left(t_1\right), & v_{x2} = v_x\left(t_2\right) \end{array}$$

etc. We also define

$$\Delta t = t_2 - t_1, \quad \Delta\mathbf{r} = \left(\Delta x, \Delta y, \Delta z\right)^{\mathrm{T}} = \mathbf{r}_2 - \mathbf{r}_1, \tag{4.15}$$

$$\Delta t' = t'_2 - t'_1, \quad \Delta\mathbf{r}' = \left(\Delta x', \Delta y', \Delta z'\right)^{\mathrm{T}} = \mathbf{r}'_2 - \mathbf{r}'_1 \tag{4.16}$$

etc. and use the simplified notation defined by

$$O^n = O\left(\left(\Delta t\right)^n\right) = \text{order of } \left(\Delta t\right)^n \text{ for } n \geq 0.$$

Due to $t' = t'(t)$ and $\alpha \mathrm{d}t = \alpha' \mathrm{d}t'$ (cf. (4.8)) we have

$$\Delta t' = t'\left(t_1 + \Delta t\right) - t'_1 = \left(dt'/dt\right)_1 \Delta t + O^2 = \frac{\alpha_1}{\alpha'_1}\Delta t + O^2$$

and therefore

$$\alpha'_1\Delta t' = \alpha_1\Delta t + O^2, \quad {\alpha'_1}^2\left(\Delta t'\right)^2 = \alpha_1^2\left(\Delta t\right)^2 + O^3, \tag{4.17}$$

and by Taylor series expansion,

$$\Delta\mathbf{r}' = \mathbf{v}'_{\mathbf{1}}\Delta t' + \frac{1}{2}\left(\Delta t'\right)^2\left(\mathrm{D}'\mathbf{v}'\right)_1 + O^3. \tag{4.18}$$

The position of S' with respect to S can be defined as $\mathbf{r}_0 = \mathbf{v}_0 t$ where

$$\mathbf{v}_0 = (v_0, 0, 0)^{\mathrm{T}}. \tag{4.19}$$

Consequently, the position, $\mathbf{r}_r$, of P with respect to S' but observed in S, and the corresponding velocity, $\mathbf{v}_r$, are given by

$$\mathbf{r}_r = \mathbf{r} - \mathbf{r}_0, \qquad \mathbf{v}_r = \mathrm{D}\mathbf{r}_r = \mathbf{v} - \mathbf{v}_0. \tag{4.20}$$

Correspondingly, the displacement $\Delta\mathbf{r}'$ with respect to S' is observed in S as a *residual displacement*

$$\Delta\mathbf{r}_r = (\Delta x_r, \Delta y_r, \Delta z_r)^{\mathrm{T}} = \Delta\mathbf{r} - \Delta\mathbf{r}_0, \tag{4.21}$$

where

$$\Delta\mathbf{r}_0 = \mathbf{v}_0 \Delta t. \tag{4.22}$$

In Newtonian kinematics we have $\Delta\mathbf{r}_r = \Delta\mathbf{r}'$, but in relativistic kinematics we obtain from (4.1), (4.21), and (4.22)

$$\begin{aligned} \Delta x_r &= \Delta x - v_0 \Delta t = \alpha_0 \Delta x', \\ \Delta y_r &= \Delta y = \Delta y', \\ \Delta z_r &= \Delta z = \Delta z'. \end{aligned} \tag{4.23}$$

For the forces acting during $[t_1, t_2]$ we can write

$$\mathbf{f} = \mathbf{f}_1 + O^1, \qquad \mathbf{f}' = \mathbf{f}'_1 + O^1,$$

thus for the corresponding work done in S' (cf. (4.16)),

$$\begin{aligned} \Delta W' &= \int_{t'_1}^{t'_2} \mathbf{f}'^{\mathrm{T}} \mathbf{v}' \mathrm{d}t' \\ &= \int_{\mathbf{r}'_1}^{\mathbf{r}'_2} \mathbf{f}'^{\mathrm{T}} \mathrm{d}\mathbf{r}' = \left(\mathbf{f}'^{\mathrm{T}}_{\mathbf{1}} + O^1\right) \Delta\mathbf{r}' \end{aligned} \tag{4.24}$$

and that done in S,

$$\Delta W = \int_{t_1}^{t_2} \mathbf{f}^{\mathrm{T}} \mathbf{v} \mathrm{d}t = \Delta W_0 + \Delta W_r \tag{4.25}$$

where (cf. (4.20))

$$\Delta W_0 = \int_{t_1}^{t_2} \mathbf{f}^{\mathrm{T}} \mathbf{v}_0 \mathrm{d}t, \tag{4.26}$$

$$\Delta W_r = \int_{t_1}^{t_2} \mathbf{f}^{\mathrm{T}} \mathbf{v}_r \mathrm{d}t = \left(\mathbf{f}_1^{\mathrm{T}} + O^1\right) \Delta\mathbf{r}_r. \tag{4.27}$$

Clearly, ΔW_0 can be considered to be the work done due to the displacement $\Delta \mathbf{r}_0$, and ΔW_r the *residual work* done. In Newtonian mechanics we have $\mathbf{v}_r = \mathbf{v}'$, $\mathbf{f} = \mathbf{f}'$, $t = t'$ and thus always $\Delta \mathrm{W}_r = \Delta \mathrm{W}'$, but the same is not true in the relativistic case.

If desired we may assume (as will be done later) that P is *instantaneously motionless* in S' at $t' = t_1'$, i.e., that

$$\mathbf{v}_1' = \mathbf{0}, \text{ thus } \mathbf{v}_1 = \mathbf{v}_0 = (v_0, 0, 0)^{\mathrm{T}} \tag{4.28}$$

(cf. (4.10)). This situation can indeed always be achieved by proper choice of S and v_0. We then have (cf. (3.2)),

$$\alpha_1 = \alpha_0, \qquad \alpha_1' = 1, \qquad v_{x1} = v_0, \qquad v_{y1} = v_{z1} = 0, \tag{4.29}$$

and, due to (4.18), (4.23), (4.24) and (4.27),

$$\Delta \mathbf{r}' = O^2, \qquad \Delta \mathbf{r}_r = O^2, \tag{4.30}$$

$$\Delta \mathbf{W}' = \mathbf{f}_1'^{\mathrm{T}} \Delta \mathbf{r}' + O^3, \;\; \Delta \mathbf{W}_r = \mathbf{f}_1^{\mathrm{T}} \Delta \mathbf{r}_r + O^3. \tag{4.31}$$

The result expressed by (4.30) and (4.31) is in line with the notion of work done as a product of force and displacement.

If (4.28) holds, the situation in S' becomes Newtonian at $t' = t_1'$, whence we may assume, as is also true in classical relativity, that

$$\mathbf{f}_1' = m_0 (\mathrm{D}'\mathbf{v}')_1 = m_0 \lim_{\Delta t' \to 0} \left(\Delta \mathbf{v}' / \Delta t' \right), \tag{4.32}$$

m_0 being some positive constant (identical to the rest mass).

More generally, the assumption (4.32) may be considered to be a consequence of a broader requirement which we call the *principle of Newtonian limit*. We may express this principle in a simple way by saying that for appropriately defined limits relativistic dynamics (i.e., dynamics compatible with the Lorentz transformation) should give the same results as Newtonian dynamics. Expressed differently and more precisely, at time instants where P is instantaneously motionless in the relevant reference frame, relativistic dynamics should , in the limit for $\Delta t' = 0$, give the same results as Newtonian dynamics.

Let us apply this principle to the ratio $\Delta W' / \Delta W_r$. According to what we have seen above (subsequently to (4.27)), we have in Newtonian dynamics

$$\lim_{\Delta t' \to 0} (\Delta W' / \Delta W_r) = 1, \tag{4.33}$$

and this even for any $\mathbf{v}_1'$. Hence, it must hold in relativistic dynamics at least if $\mathbf{v}_1' = \mathbf{0}$. Since $\Delta W'$ and ΔW_r are O^2 (cf. (4.30) and (4.31)) we may thus equivalently say that for $\mathbf{v}_1' = \mathbf{0}$ we have

$$\lim_{\Delta t' \to 0} \left(\frac{\Delta W'}{(\Delta t')^2} \right) = \lim_{\Delta t' \to 0} \left(\frac{\Delta W_r}{(\Delta t')^2} \right). \tag{4.34}$$

Since that last result must hold without restriction it must be true for any $\mathbf{f}_1'$. Due to (4.18), (4.23), (4.31), and (4.32) an alternative way to express it is to require

$$\mathbf{f}_1'^{\mathrm{T}} \mathbf{f}_1' = (\alpha_0 f_{x1}, f_{y1}, f_{z1})\, \mathbf{f}_1' \qquad \forall \mathbf{f}_1' \tag{4.35}$$

or else,

$$(\alpha_0 f_{x1} - f_{x1}') f_{x1}' + \big(f_{y1} - f_{y1}' \big) f_{y1}' + (f_{z1} - f_{z1}') f_{z1}' = 0 \quad \forall \mathbf{f}_1'. \tag{4.36}$$

For this it is necessary and sufficient that

$$\alpha_0 f_{x1} = f'_{x1}, \quad f_{y1} = f'_{y1}, \quad f_{z1} = f'_{z1}. \tag{4.37}$$

Due to (4.28) we have $v_{y1} = v_{z1} = 0$ and thus, in view of (4.11) to (4.13) and (4.29),

$$(\mathrm{D}' v'_x)_1 = \left(\mathrm{D}\frac{v_x}{\alpha}\right)_1,$$

$$\left(\mathrm{D}' v'_y\right)_1 = \frac{1}{\alpha_1}\left(\mathrm{D}\frac{v_x}{\alpha}\right)_1, \qquad (\mathrm{D}' v'_z)_1 = \frac{1}{\alpha_1}\left(\mathrm{D}\frac{v_z}{\alpha}\right)_1.$$

Hence, taking into account $\alpha_0 = \alpha_1$ (cf. (4.29)), (4,32) and (4.37) yield

$$\mathbf{f}_1 = \frac{m_0}{\alpha_1}\left(\mathrm{D}\frac{\mathbf{v}}{\alpha}\right)_1. \tag{4.38}$$

Clearly, (4.38) implies the same type of relations for all three components f_{x1}, f_{y1}, and f_{z1}, and this despite the differences in (4.37). Thus the orientation originally adopted for $\mathbf{v_0}$ is totally irrelevant. But t_1 is arbitrary. Hence we can finally indeed replace (4.38) by

$$\mathbf{f} = \frac{m_0}{\alpha}\mathrm{D}\left(\frac{1}{\alpha}\mathbf{v}\right), \tag{4.39}$$

and therefore,

$$\mathbf{f} = \sqrt{m}\mathrm{D}\left(\sqrt{m}\mathbf{v}\right) = \frac{1}{2}\left(m\mathrm{D}\mathbf{v} + \mathrm{D}(m\mathbf{v})\right), \ \ m = \frac{m_0}{\alpha^2}, \tag{4.40}$$

confirming (3.6) and (3.9).

In view of Einstein's second postulate, a relation of exactly the same type can of course be written for $\mathbf{f}'$, $\mathbf{v}'$, and α', i.e.,

$$\mathbf{f}' = \frac{m_0}{\alpha'}\mathrm{D}'\left(\frac{1}{\alpha'}\mathbf{v}'\right).$$

In particular, if $v_y \equiv v_z \equiv 0$, i.e, if all phenomena take place exclusively along the x and x' axes, we have $\alpha_x = \alpha$, $\alpha'_x = \alpha'$, and we then obtain, using (4.14),

$$\alpha\mathbf{f} = \alpha'\mathbf{f}'. \tag{4.41}$$

More generally, we have

$$f_x = \frac{1}{\alpha_0}\left(\frac{\beta_0}{c}\mathbf{v}'^{\mathrm{T}}\mathbf{f}' + f'_x\right), \qquad f_y = f'_y, \qquad f_z = f'_z, \tag{4.42}$$

$$\mathbf{v}^{\mathrm{T}}\mathbf{f} = \frac{1}{\alpha_0}\left(\mathbf{v}'^{\mathrm{T}}\mathbf{f}' + c\beta_0 f'_x\right) \tag{4.43}$$

where all quantities are as defined before and used so far. This result can either be verified directly by means of (4.1) etc. or by recalling that a proper four-vector such as the Minkowski force (cf. (3.11)) transforms from S' to S by premultiplying the one in S' with the matrix

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\alpha_0} & 0 & 0 & \frac{\beta_0}{\alpha_0} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\beta_0}{\alpha_0} & 0 & 0 & \frac{1}{\alpha_0} \end{pmatrix}, \tag{4.44}$$

(whose inverse M^{-1} is obtained by simply changing the sign of β_0). The result (4.41) can of course be shown to be a special case of the present general one.

Observe that the entire derivation, from (4.32) to (4.38), involves $\mathbf{v}'$ only via $(\mathrm{D}'\mathbf{v}')_1$. This is of interest because it stresses the role of that quantity as an asymptotic measure. On the other hand, in order to obtain the classical relativistic expression for the force, one would have to replace the right-hand side of (4.33), as can be concluded by inspecting (4.34), by $1/\alpha_0$, which would amount to a somewhat surprising requirement.

So far we have taken it for granted that m_0 is constant. Nevertheless there may by some interest to allow also for rest masses m_0 that, although independent of $\mathbf{v}$, are dependent on t in some other way than via $\mathbf{v}$. Let us assume that in that case, (4.32) is of the form

$$\mathbf{f}_1' = \sqrt{m_0}\,(D'\sqrt{m_0}\mathbf{v}')_1 \,.$$

It is easily verified that under this assumption the above process for arriving at (4.40) would still be valid, but (4.39) would then in general no longer hold,

5 Modification of Newton's third law and conservation of momentum

In classical relativity, Newton's third law remains untouched. This is true to the point that it is common practice not even to mention that keeping it unchanged amounts to making a corresponding definite additional assumption. If we adopt the alternative viewpoint, however, such an assumption is nowhere justified, i.e., we have to modify not only Newton's second but also his third law.

Consider indeed action and reaction between two particles, P_1 and P_2, the subscripts 1 and 2 being consistently used in this section in order to distinguish between quantities referring either to P_1 or P_2, respectively. We thus assume P_1 and P_2 to be travelling with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Let there be an interaction between P_1 and P_2 and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the resulting forces acting upon P_1 and P_2, respectively. We first assume $\mathbf{f_1}, \mathbf{f_2}, \mathbf{v_1}$, and $\mathbf{v}_2$ to be all located on the same straight line, which we may assume to be the x axis of S, thus the x' axis of S' (defined as before); for S' we correspondingly have to consider $\mathbf{f_1'}, \mathbf{f_2'}, \mathbf{v_1'}$, and $\mathbf{v}_2'$ (and similarly primed quantities as used hereafter). Defining

$$\beta_1 = v_{x1}/c, \quad \beta_1' = v_{x1}'/c, \quad \beta_2 = v_{x2}/c, \quad \beta_2' = v_{x2}'/c,$$
$$\alpha_i = \sqrt{1-{\beta_i}^2}, \quad \alpha_i' = \sqrt{1-\beta_i'^2}, \quad i=1,2 \quad , \tag{5.1}$$

we can write (cf.(4.10)),

$$\beta_1' = \frac{\beta_1-\beta_0}{1-\beta_1\beta_0}, \quad \beta_2' = \frac{\beta_2-\beta_0}{1-\beta_2\beta_0}, \quad \beta_0 = v_0/c\,. \tag{5.2}$$

Assume P_1 and P_2 to be touching each other at $t=t_0$, which implies $x_1(t_0)=x_2(t_0)$ and thus (cf. (4.1)) $x_1'(t_0')=x_2'(t_0'),\ t_0'=t_1'(t_0)=t_2'(t_0)$. To simplify the writing we add a subscript zero to specify evaluations at $t=t_0$, thus at $t'=t_0'$ (except that v_0 and thus β_0 have the same meaning as so far). If $\beta_{10}=-\beta_{20}$ we adopt $S'=S$. If $\beta_{10}\neq-\beta_{20}$, choosing β_0 such that

$$\beta_0^2-2b\beta_0+1=0, \qquad b=(1+\beta_{10}\beta_{20})/(\beta_{10}+\beta_{20}),$$

which, due to

$$b^2 = 1 + \left(\frac{\alpha_{10}\alpha_{20}}{\beta_{10} + \beta_{20}} \right)^2,$$

always yields real values for β_0, we obtain, as can be shown (cf. (5.2)),

$$\beta'_{10} = -\beta'_{20}, \text{ thus } \alpha'_{10} = \alpha'_{20}. \tag{5.3}$$

Note that the product of the two choices for β_0 is equal to 1, i.e., we must select that choice for which $|\beta_0| < 1$.

Due to the assumption concerning the orientation of the velocities and forces and the reference frames we may make use of (4.41) and thus write in particular,

$$\alpha_{10}\mathbf{f}_{10} = \alpha'_{10}\mathbf{f}'_{10}, \quad \alpha_{20}\mathbf{f}_{20} = \alpha'_{20}\mathbf{f}'_{20}. \tag{5.4}$$

But due to the symmetry of the velocities in S' we must have $\mathbf{f}'_{10} = -\mathbf{f}'_{20}$. Hence, (5.3) and (5.4) yield

$$\alpha_{10}\mathbf{f}_{10} = -\alpha_{20}\mathbf{f}_{20}. \tag{5.5}$$

Furthermore if, while maintaining the assumption concerning the alignment of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{f}_1$, $\mathbf{f}_2$, we abandon the assumption concerning the coincidence of that common direction with the one of the x and x' axes, an orientation argument as used in Section 4.2 (following (4.38)) shows that (5.5) remains valid.

Consider now the general case of the relationship between action and reaction. In view of (5.5) it is clear that it could not be $\mathbf{f}_1 = -\mathbf{f}_2$ as in the classical theory. For modifying Newton's third law, however, the simplest way compatible with (5.5) is to write

$$\alpha_1\mathbf{f}_1 = -\alpha_2\mathbf{f}_2 \tag{5.6}$$

where α_1 and α_2 are defined according to the subscript convention mentioned at the beginning of this section (cf. (5.1)).

Next, let m_{10} and m_{20} be the rest masses of P_1 and P_2. It follows from (4.39) and (5.6) that

$$\mathrm{D}(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{0}$$

where

$$\mathbf{p}_1 = m_{10}\mathbf{v}_1/\alpha_1 \quad \text{and} \quad \mathbf{p}_2 = m_{20}\mathbf{v}_2/\alpha_2$$

are the momenta as mentioned in Section 3. More generally, if there are n particles P_1 to P_n and if forces $\mathbf{f}_{\nu 1}$ to $\mathbf{f}_{\nu k}$ are acting upon P_ν, which is moving with velocity $\mathbf{v}_\nu$, we have

$$\mathrm{D}\mathbf{p}_\nu = \alpha_\nu \left(\mathbf{f}_{\nu 1} + \cdots + \mathbf{f}_{\nu k} \right) \tag{5.7}$$

where

$$\mathbf{p}_\nu = m_{\nu 0}\mathbf{v}_\nu/\alpha_\nu,$$
$$\alpha_\nu = \sqrt{1 - \beta_\nu^2}, \quad \beta_\nu = v_\nu/c, \quad v_\nu^2 = \mathbf{v}_\nu^{\mathrm{T}}\mathbf{v}_\nu.$$

Summing (5.7) over all particles, there will be pairwise cancellations in the right-hand side due to relations of the type of (5.6), yielding

$$\mathrm{D}(\mathbf{p}_1 + \cdots + \mathbf{p}_n) = \mathbf{0}. \tag{5.8}$$

Hence, conservation of momentum holds exactly as in classical relativity. Vice-versa, if in the alternative theory we require (5.8) to apply in all cases, we are forced to modify Newton's third law as shown in (5.6).

6 Moving particles in fields – forces and energy

Assume that a field (electromagnetic, gravitational etc.) is exerting a force $\mathbf{f}$ upon a particle P travelling with velocity $\mathbf{v}$, that the equipment, Eq (referred to hereafter by a subscript 1), producing the field is at rest ($\mathbf{v}_1 = \mathbf{0}$), and that the reaction upon Eq is $\mathbf{f}_1$. Assume furthermore that all relevant distances are sufficiently small so that we can state (cf. (5.6)),

$$\alpha \mathbf{f} = -\alpha_1 \mathbf{f}_1 = -\mathbf{f}_1, \tag{6.1}$$

with α and α_1 as in (3.2) and as used in Section 5 (except that Eq assumes the former role of P_1, whence $\alpha_1 = 1$). If $\mathbf{f}_1$ is independent of the velocity of P, it must be equal to the force existing for $\mathbf{v} = \mathbf{0}$. Hence, we then conclude from (6.1) that

$$\mathbf{f} = \mathbf{f}_0/\alpha \tag{6.2}$$

where $\mathbf{f}_0$ is that force that the field exerts upon P if $\mathbf{v} = \mathbf{0}$, and thus is equal to $-\mathbf{f}_1$, i. e., $\mathbf{f}_0$ is the force as used in all classical expressions. We have e. g. $\mathbf{f}_0 = q\mathbf{E}$ if q is a charge and $\mathbf{E}$ the electric field, and correspondingly in the case of a gravitational field.

For an electromagnetic field this is confirmed and in fact extended to include the Lorentz force if one proceeds as is classically done (see e.g. Section 7.7 of [19]), i.e., if, at the instant considered, one derives $\mathbf{f}$ by Lorentz transforming the force $q\mathbf{E}'$ observed in an appropriate other reference system S'. For seeing this, let us first recall that the force $\mathbf{f}$ considered here is identical to the corresponding part of the Minkowski four-vector (cf. Section 3) and that the electric and magnetic fields are known to transform between S' and S (both used as so far) according to

$$E'_x = E_x, \quad E'_y = \frac{1}{\alpha_0}\left(E_y - v_0 \mu H_z\right), \quad E'_z = \frac{1}{\alpha_0}\left(E_z + v_0 \mu H_y\right), \tag{6.3}$$

$$H'_x = H_x, \quad H'_y = \frac{1}{\alpha_0}\left(H_y + v_0 \varepsilon E_z\right), \quad H'_z = \frac{1}{\alpha_0}\left(H_z - v_0 \varepsilon E_y\right), \tag{6.4}$$

primed field quantities referring again to S' and unprimed ones to S.

Consider then a charge q that is instantaneously motionless in S', i.e., we assume the orientation of S and S' and the choice of $\mathbf{v}_0$ (cf. (4.19)) to be such that at the time instant considered we have $\mathbf{v}' = \mathbf{0}$ thus $\mathbf{v} = \mathbf{v}_0$ (cf. (4.10)) and therefore also $\alpha_0 = \alpha$. The force $\mathbf{f}'$ acting on q is then given by

$$\mathbf{f}' = q\mathbf{E}', \quad \mathbf{E}' = \left(E'_x, E'_y, E'_z\right)^{\mathrm{T}}.$$

From (4.42) and (6.3) we thus obtain, making use of the magnetic induction defined by $\mathbf{B} = (B_x, B_y, B_z)^{\mathrm{T}} = \mu\mathbf{H}$,

$$f_x = q\frac{1}{\alpha}E_x, \quad f_y = q\frac{1}{\alpha}\left(E_y - v_0 B_z\right), \quad f_z = q\frac{1}{\alpha}\left(E_z - v_0 B_y\right).$$

This result can indeed be written in the form (6.2) with

$$\mathbf{f}_0 = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{6.5}$$

In the way the result is presented in (6.2) and (6.5) it is expressed by only using quantities defined in S and general vector notation. Hence, it is valid without restriction.

On the other hand, a force $\mathbf{f}$ acting on a particle of rest mass m_0 also satisfies (4.39). We thus derive from (6.2),

$$m_0 \mathrm{D}\left(\frac{1}{\alpha}\mathbf{v}\right) = \mathbf{f}_0. \tag{6.6}$$

We conclude from (6.6) that the alternative theory leads to exactly the same dynamic behavior of a particle in a field as classical relativity theory.

This however does not include energy. Consider indeed a relation from classical electrodynamics such as

$$-\rho\mathbf{v}^{\mathrm{T}}\mathbf{E} = \frac{1}{2}\mathrm{D}\left(\varepsilon\mathbf{E}^{\mathrm{T}}\mathbf{E} + \mu\mathbf{H}^{\mathrm{T}}\mathbf{H}\right) + \mathrm{div}\left(\mathbf{E}\times\mathbf{H}\right)$$

(written in standard notation), where $\rho = \varepsilon\mathrm{div}\mathbf{E}$ is the charge density and ε and μ are assumed constant (say $\varepsilon = \varepsilon_0,\ \mu = \mu_0$). In order to obtain a proper expression for the energy supplied, the alternative theory obviously requires, as has been shown in this section, to multiply the classical force density $\rho\mathbf{E}$ in the left-hand side by $1/\alpha$, and the right-hand side thus has to be multiplied in the same way. We consider here only the case of a constant α (see also Section 7). Clearly, the field energy density then turns out to be

$$w = \frac{1}{2\alpha}\left(\varepsilon\mathbf{E}^{\mathrm{T}}\mathbf{E} + \mu\mathbf{H}^{\mathrm{T}}\mathbf{H}\right) \tag{6.7}$$

and the Poynting vector similarly becomes $(\mathbf{E}\times\mathbf{H})/\alpha$. The classical expression $(\varepsilon\mathbf{E}^{\mathrm{T}}\mathbf{E} + \mu\mathbf{H}^{\mathrm{T}}\mathbf{H})/2$ is then something like an apparent energy density.

Note that in this and the subsequent sections we use the letters W and w for designating energies and energy densities, and this in order to facilitate distinguishing between energies and electric fieds. Also note that if instead of a charge q we consider a charge density ρ travelling with velocity $\mathbf{v}$ we can replace (6.5) by,

$$\hat{\mathbf{f}} = \hat{\mathbf{f}}_0/\alpha, \quad \hat{\mathbf{f}}_0 = \rho\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

where $\hat{\mathbf{f}}$ and $\hat{\mathbf{f}}_0$ are force densities, i.e., forces per unit volume.

7 Electromagnetic field moving with constant velocity

We define S and S' as so far. Let us first consider a field that in S' is electrostatic, thus at rest, and given there by $\mathbf{E}'$. Clearly, $\mathbf{E}'$ is independent of t'. Although for the corresponding magnetic field we have $\mathbf{H}' = \mathbf{0}$, the field in S comprises both $\mathbf{E}$ and $\mathbf{H}$. Since we assume ε and μ constant we need not consider $\mathbf{D}$ and $\mathbf{B}$. In S, the field is moving with constant velocity $\mathbf{v}$. We have $\mathbf{v} = \mathbf{v}_0$ (cf. (4.10)) and therefore,

$$v = v_0, \quad \alpha = \alpha_0, \quad \beta = \beta_0, \tag{7.1}$$

the quantities thus involved being defined as so far (except that we assume v to take the sign of v_0). From (6.3), (6.4), and (7.1) we deduce, for any $\mathbf{E}'$ and $\mathbf{H}'$,

$$\begin{aligned} &E_x = E'_x, \quad E_y = \frac{1}{\alpha}\left(E'_y + v\mu H'_z\right), \quad E_z = \frac{1}{\alpha}\left(E'_z - v\mu H'_y\right), \\ &H_x = H'_x, \quad H_y = \frac{1}{\alpha}\left(H'_y - v\varepsilon E'_z\right), \quad H_z = \frac{1}{\alpha}\left(H'_z + v\varepsilon E'_y\right). \end{aligned} \tag{7.2}$$

For the field energy density defined by (6.7) we thus find for $\mathbf{H}' = \mathbf{0}$,

$$w = \frac{1}{\alpha} w' + \frac{\beta^2}{\alpha^3} w_0', \tag{7.3}$$

where

$$w' = \frac{1}{2}\varepsilon \left({E_x'}^2 + {E_y'}^2 + {E_z'}^2 \right), \quad w_0' = \varepsilon \left({E_y'}^2 + {E_z'}^2 \right), \tag{7.4}$$

w' being the field energy density in S'. Due to the electrostatic hypothesis, the right-hand side of (7.3) is independent of t'.

Let then

$$\mathrm{d}V = \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z, \quad \mathrm{d}V' = \mathrm{d}x' \cdot \mathrm{d}y' \cdot \mathrm{d}z'$$

be the elementary volumes in S and S', respectively. As is known, it follows from (4.1) that for any fixed t,

$$\mathrm{d}x = \alpha \mathrm{d}x', \quad \mathrm{d}y = \mathrm{d}y', \quad \mathrm{d}z = \mathrm{d}z',$$

so that $\mathrm{d}V = \alpha \mathrm{d}V'$. We thus obtain from (7.3),

$$w \mathrm{d}V = w' \mathrm{d}V' + \frac{\beta^2}{\alpha^2} w_0' \mathrm{d}V'. \tag{7.5}$$

This expression is precisely of the form determined by (3.8) and (3.10), with E, E_i, and E_0 replaced by $w\mathrm{d}V$, $w'\mathrm{d}V'$, and $w_0'\mathrm{d}V'$, respectively, and such that no strict relationship exists between w' and w_0'. We may also integrate (7.5) over the entire volume, which leads to $W = W' + W_k$ where

$$W = \int_V w dV, \quad W' = \int_{V'} w' dV', \quad W_k = \frac{\beta^2}{\alpha^2} \int_{V'} w_0' dV'$$

Since w' and w_0' are independent of t', W' and W_k may be considered to be evaluated at a constant t'. Furthermore, V and V' may be the entire space.

In all cases we thus obviously obtain full agreement with the results developed in the preceding sections. The decomposition into rest energy and kinetic energy encountered in classical relativity theory does not offer a similarly elegant interpretation. This is remarkable since Maxwell's equations are known to be inherently compatible with the Lorentz transformation.

The second expression (7.4) can also be written as

$$w_0' = \varepsilon \mathbf{E}'^{\mathrm{T}} \mathbf{E}' - \varepsilon {E_x'}^2.$$

But due to $\mathbf{v} = \mathbf{v}_0$ and $v = v_0$ we have (cf. (7.1)) $v E_x' = \mathbf{v}^{\mathrm{T}} \mathbf{E}'$. Hence, (7.5) can be expressed in the more general form

$$w \mathrm{d}V = \left(w' + \frac{1}{2\alpha^2} \mathbf{v}^{\mathrm{T}} \mathbf{m}_0 \mathbf{v} \right) \mathrm{d}V', \tag{7.6}$$

where

$$\mathbf{m}_0 = \frac{2\varepsilon}{c^2} \left(\left(\mathbf{E}'^{\mathrm{T}} \mathbf{E}' \right) \mathbf{1} - \mathbf{E}' \mathbf{E}'^{\mathrm{T}} \right), \tag{7.7}$$

$\mathbf{1}$ being again the unit matrix of order 3. These expressions hold for any orientation of $\mathbf{v}$ and have the form of the matrix case discussed in relation with (3.14).

Consider now a general electromagnetic field. Substitution of (7.2) in (6.7) yields, after some calculation, the more general expression

$$w\mathrm{d}V = \left(w' + \frac{1}{2\alpha^2}\mathbf{v}^{\mathrm{T}}\mathbf{m}_0\mathbf{v} + \frac{2}{c^2\alpha^2}\mathbf{v}^{\mathrm{T}}\mathbf{S}'\right)\mathrm{d}V' \tag{7.8}$$

where

$$w' = \frac{1}{2}\left(\varepsilon\mathbf{E}'^{\mathrm{T}}\mathbf{E}' + \mu\mathbf{H}'^{\mathrm{T}}\mathbf{H}'\right), \quad \mathbf{S}' = \mathbf{E}' \times \mathbf{H}', \tag{7.9}$$

$$\mathbf{m}_0 = \frac{2}{c^2}\left(\left(\varepsilon\mathbf{E}'^{\mathrm{T}}\mathbf{E}' + \mu\mathbf{H}'^{\mathrm{T}}\mathbf{H}'\right)\mathbf{1} - \left(\varepsilon\mathbf{E}'\mathbf{E}'^{\mathrm{T}} + \mu\mathbf{H}'\mathbf{H}'^{\mathrm{T}}\right)\right). \tag{7.10}$$

Compared to (7.6), (7.7) and the first equation (7.4), the changes in w' and $\mathbf{m}_0$ are as immediately expected. There is however an additional term that is proportional to the component of the Poynting vector in S' in the direction of $\mathbf{v}$, thus of the corresponding momentum density in S' of the electromagnetic field. This additional term vanishes in particular if at the point P' (thus at the position and the time instant) under consideration in S' the field has zero momentum, i.e., in a sense, is there instantaneously motionless. If we then consider (6.7) but apply it to S' instead of S, it reduces to the first equality in (7.4). Hence, invoking again a principle akin to that of Newtonian limit (but now applied in an electromagnetic sense) we can state that w' is, under the present assumptions, indeed equal to the field energy density at P'.

8 The Bertozzi experiment

In 1964 Bertozzi [26] has published a few results of experiments aimed at determining the kinetic energy of fast electrons. Expressed in terms of a notation more suitable for our purpose, he had determined values of $\beta_b^2 = 1 - \alpha_b^2$, thus of v_b^2/c^2, in terms of what we designate here by

$$\gamma = qu_0/m_0c^2,$$

q being the electron charge, u_0 the voltage traversed by the electrons in the accelerator, m_0 the electron rest mass, v_b the velocity reached by the electron just before hitting the intended target, and c the speed of light. According to classical relativity theory α_b is simply related to the other quantities by

$$\frac{m_0c^2}{\alpha_b} - m_0c^2 = qu_0, \tag{8.1}$$

yielding

$$\alpha_b^2 = 1 - \beta_b^2 = 1\big/(1+\gamma)^2, \quad \text{with} \quad \beta_b = v_b/c. \tag{8.2}$$

Since according to the alternative theory the dynamic behavior of particles in fields is exactly the same as that predicted by classical relativity, the result expressed by (8.2) is valid in exactly the same way in both theories (contrary to what had been stated in [21]).

Let E_k be the kinetic energy of the electron travelling at v_b. According to classical relativity we have $E_k = qu_0$, whence in [26] γ had indeed been set equal to E_k/m_0c^2. According to the alternative theory, however, $E_k = m_0v_b^2/2\alpha_b^2$, which in view of (8.2) yields

$$E_k = qu_0(1+\gamma/2) = m_0c^2\gamma(1+\gamma/2), \tag{8.3}$$

and this discrepancy was at the origin of the confusion in [21].

For testing the validity of equating E_k with qu_0, Bertozzi had also, in two cases, determined the heat, E_h, generated by an electron hitting the target. Define $\delta = E_h/qu_0$. Since there are some unavoidable losses (e.g. by X-ray generation) we have $E_h < E_k$, thus $\delta < E_k/qu_0$. Hence, $\delta < 1$ if we had indeed $E_k = qu_0$. In both tests (1.5 MeV and 4.5 MeV), the measured result was $\delta = 1.067$, which is compatible with $\delta < 1$ only if one makes allowance for the low accuracy (10 % as stated in [26]) of the experiment.

For examining the situation according to the alternative theory, let f be the braking force acting upon the electron in the target and let f_f be the corresponding friction force acting on the target, which we may assume to be at rest. Due to the alternative form of Newton's third law we have

$$\alpha f = f_f, \quad \alpha = \sqrt{1-\beta^2}, \quad \beta = v/c,$$

i.e., in view of (4.39), $f_f = m_0 \mathrm{D}\,(v/\alpha)$, v being the electron velocity at the considered time instant. Hence (cf. (3.5)), the heat power generated by friction is $vf_f = \mathrm{D}\,(m_0c^2/\alpha)$, and the heat generated by the braking process is therefore given by

$$\left[\frac{m_0c^2}{\alpha}\right]_{v=0}^{v=v_b} = \frac{m_0c^2}{\alpha_b} - m_0c^2 = qu_0,$$

the second equality being indeed a consequence of (8.1). The excess energy $E_k - qu_0$ (cf. (8.3)) will be converted partly into radiation and partly into (practically unmeasurable) kinetic energy of the target and its supporting body, but, due to secondary processes, a small part will again turn up in form of heat, thus causing $E_h > qu_0$, i.e., $\delta > 1$. This agrees with the tendency exhibited by the outcome of the experiment, i.e., without having to invoke an insufficient experimental accuracy.

9 The matrix case

In Sections 3 and 7 we have briefly mentioned the possibility of having objects whose mass cannot simply be described by a scalar, as is usually assumed, but only by a matrix. Let us examine a few details about the properties of such a (particle-like) object P, which we thus assume to be characterized, at vanishing velocity, by a constant (i.e. time-independent) symmetric positive definite matrix $\mathbf{m}_0$ called *matrix rest mass*.

We first have to reexamine the derivation presented in Section 4.2, i.e., for m_0 replaced by $\mathbf{m}_0$ and thus (4.32) by

$$\mathbf{f}_1' = \mathbf{m}_0\,(\mathrm{D}'\mathbf{v}')_1\,. \tag{9.1}$$

However, if $\mathbf{m}_0$ is a fully occupied matrix we cannot take it for granted anymore that the value ΔW_r defined by (4.25) and (4.26) is a proper estimate of $\Delta W'$. Indeed if $\mathbf{m}_0$ is as just stated it is not possible to sufficiently separate the phenomena in the $x, y, \text{and } z$ directions, while due to (4.19) the corrective term (4.26) singles out the x direction. We therefore first assume that the directions of $x', y', \text{and } z'$ coincide with the principal axes of $\mathbf{m}_0$, i.e., that $\mathbf{m}_0$ is diagonal, thus $\mathbf{m}_0 = \mathrm{diag}\,(m_1, m_2, m_3)$, and we may then again claim (4.34) to hold.

However, in view of (9.1), $(\mathrm{D}'\mathbf{v}')_1$ appearing in (4.18) is no longer equal to a scalar (formerly: $1/m_0$, cf. (4.32)) times $\mathbf{f}_1'$, so that the two right vector factors $\mathbf{f}_1'$ in (4.35) have to be replaced by the original factor $(\mathrm{D}'\mathbf{v}')_1$. We have to proceed correspondingly for the three right scalar factors in (4.36) where, due to (9.1), $\forall \mathbf{f}'_1$ implies $\forall \left(\mathrm{D}'\mathbf{v}'\right)_1$ so that (4.37) is still valid. From this result, we can follow again the same steps as in Section 4.2, using for $\mathbf{m}_0$ the property of being diagonal. One finds this way that (4.38) has to be replaced by

$$\mathbf{f}_1 = \frac{1}{\alpha_1}\mathbf{m}_0 \left(\mathrm{D}\frac{\mathbf{v}}{\alpha}\right)_1$$

and thus (4.39) and (4.40) by (3.13), with $\mathbf{m}$ given by (3.12). Observe that for vanishing velocity, P is again fully characterized by $\mathbf{m}_0$; this is in agreement with Einstein's first postulate but holds in the strict form just mentioned only because of the specific choices of $x', y', \text{and } z'$ with respect to $x, y, \text{and } z$ (contrary to what is the case for the earlier considered scalar m_0).

We now drop the assumption for $\mathbf{m}_0$ to be diagonal. Let P be travelling in S with velocity $\mathbf{v}$ and subjected there to a force $\mathbf{f}$, $\mathbf{m}_0$ being the matrix mass for vanishing velocity, i.e. the matrix rest mass. Let S_0 be a further reference system that is in a fixed, yet rotated position with respect to S. There thus exists a constant orthogonal matrix $\mathbf{U}$ such that $\mathbf{Uf}$ and $\mathbf{Uv}$ are the force and the velocity in S_0. Since for vanishing velocity we have $\mathbf{f} = \mathbf{m}_0\mathrm{D}\mathbf{v}$, the matrix rest mass in S_0 is $\mathbf{U}\mathbf{m}_0\mathbf{U}^{\mathrm{T}}$.

On the other hand, since $\mathbf{m}_0$ is constant we may indeed choose the constant matrix $\mathbf{U}$ such that $\mathbf{U}\mathbf{m}_0\mathbf{U}^{\mathrm{T}}$ is diagonal. Hence, using the result obtained above we may write for S_0,

$$\mathbf{Uf} = \frac{1}{\alpha}\left(\mathbf{U}\mathbf{m}_0\mathbf{U}^{\mathrm{T}}\right)\mathrm{D}\left(\frac{1}{\alpha}\mathbf{Uv}\right). \tag{9.2}$$

This implies again (3.13) to hold, with $\mathbf{m}$ as given by (3.12).

This neat result confirms the appropriateness of extending the scalar to the matrix case in the simple way we have chosen in Section 3. There are, however, also discrepancies, and this concerns in particular the classical concept of transforming four-vectors between S ans S' by premultiplying them with the matrix $\mathbf{M}$ (cf. (4.44)) or its inverse. The classical proof for justifying this approach in the case of the Minkowski force is indeed not applicable because $\mathbf{M}$ and $\mathbf{m}_0$ will in general not commute. Nevertheless, quite simple proofs can be obtained by making use of elegant expressions such as

$$\begin{aligned} 1-\beta_0\beta_x &= \alpha_0\alpha/\alpha', \quad 1+\beta_0\beta'_x = \alpha_0\alpha'/\alpha \\ 1-\beta_x\beta'_x &= \alpha_x\alpha'_x/\alpha_0, \quad \alpha'/\alpha = \alpha'_x/\alpha_x \end{aligned} \tag{9.3}$$

which can indeed be shown to follow from (4.10) (cf. also (4.8)) and where all notation used is as defined earlier.

Let us thus derive the equivalent of (4.42). We find, using (3.13), (4.8) (4.10), and (9.3),

$$\mathbf{f} = \mathbf{m}_0\frac{1}{\alpha}\mathrm{D}\left(\frac{\mathbf{v}}{\alpha}\right) = \mathbf{m}_0\frac{1}{\alpha'}\mathrm{D}'\left(\frac{1}{\alpha'}\left((v'_x+v_0)/\alpha_0,\ v'_y,\ v'_z\right)^{\mathrm{T}}\right). \tag{9.4}$$

This yields, applying the equivalent of (3.5) for S',

$$\mathbf{f} = \mathbf{m}_0\frac{1}{\alpha'}\mathrm{D}'\left(\frac{1}{\alpha'}\begin{pmatrix} v'_x/\alpha_0 \\ v'_y \\ v'_z \end{pmatrix}\right) + \mathbf{m}_0\frac{\beta_0}{2c\alpha_0}\begin{pmatrix} \mathrm{D}'\left(v'/\alpha'\right)^2 \\ 0 \\ 0 \end{pmatrix}.$$

That expression cannot usually be reduced to the form of (4.42), which is the one that would be obtained by applying the standard procedure based on (4.44). For $\mathbf{m}_0$ diagonal, say for $\mathbf{m}_0 = \mathrm{diag}(m_1, m_2, m_3)$ one finds

$$f_x = \frac{1}{\alpha_0}\left(f'_x + m_1\frac{\beta_0}{c\alpha'}\mathbf{v}'^{\mathrm{T}}\mathrm{D}'\left(\frac{\mathbf{v}'}{\alpha'}\right)\right), \quad f_y = f'_y, \quad f_z = f'_z,$$

but even that expression reduces to (4.42) only if $m_1 = m_2 = m_3$, which is equivalent to the scalar case.

A comment similar to that at the end of Section 4.2 can also be made in the present context. In particular, an object as we are considering here may be subject to influences other than forces (cf. Section 7), which can bring about an implicit time dependence.

10 Conclusions

Proper characterization of losslessness in Kirchhoff circuits requires the defining equations of elements such as inductances to have a very specific form, which in turn can be recognized to be of fundamental physical importance. Surprisingly, this form is not respected by the classical relativistic expression for the force acting on a particle. However, requiring the relativistic laws to become coincident with the Newtonian laws when approaching a time instant where the particle is instantaneously motionless, and applying this principle of Newtonian limit in particular to the force and to the work done, one finds an expression for the force, thus for the relativistic formulation of Newton's second law, that is in perfect agreement with that to be expected from the theory of nonlinear Kirchhoff circuits. This difference comes about essentially by putting prime emphasis on energy aspects (losslessness, work done, etc,) rather than on momentum, as in classical relativity. The alternative way of modifying Newton's second law is shwon to require also his third law, which in the classical theory remains untouched, and as a consequence the expressions for forces in fields to be changed correspondingly. These changes combined imply that conservation of momentum and dynamics of particles remain exactly as in classical relativity, the latter aspect being confirmed and extended by appropriately interpreting the classical application of the Lorentz transformation to Maxwell's equations.

The rest energy of particles appears as an arbitrary integration constant, and its value thus remains fully compatible with the classical relativistic expression. Definite differences between the classical and the alternative theory become visible for the kinetic energy, which for the latter theory is rational in the velocity while it is known to be irrational for the former. For an electric field that is static with respect to a reference frame moving itself uniformly with respect to the observer, the kinetic term in the field energy agrees fully and directly with the alternative expression for kinetic energy, thus contrary to the classical expression. The few measured results published by Bertozzi about the kinetic energy of fast electrons favor the alternative theory, but the inaccuracy of 10% mentioned by the author is too large to allow definite conclusions to be drawn.

Some further-reaching ideas could be developed that concern the nature of particles and are compatible with the present results. A very brief outline of this can be found at the end of [21], although some aspects of that paper are revised or extended by the present one.

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Alfred Fettweis was born in Eupen, Belgium, on 27 November 1926. He studied electrical engineering at Université Catholique de Louvain (1946 - 51) and received from there the degree ”Docteur en sciences appliquées” in 1963. He worked for International Telephone and Telegraph Corporation in Belgium (1951 - 54, 1956 - 63) and the USA (1954 - 56), was a professor of theoretical electricity at Technical University of Eindhoven, Netherlands (1963 - 67), has been professor of communications engineering at Ruhr-Universität Bochum, Germany (since

1967, emeritus professor since 1992) and was a distinguished visiting professor at Notre Dame University, USA (1994 - 96). He received honorary doctorates from Linköping 1986, Mons 1988, Leuven 1988, Budapest 1995, and several awards, among which: Prix Acta Technica Belgica 1963, Prix Montefiore 1980, VDE-Ehrenring 1984, ITG Karl-Küpfmüller-Preis 1988, Eduard Rhein Basic Research Award 1993, IEEE Centennial (1984) and Millenium (1999) Medal, and Darlington Award 1980, Technical Achievement Award 1988, Van Valkenburg (Society) Award 2001, (first) Vitold Belevitch Award 2003 (all 4 from IEEE Circuits and Systems Society).