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Relative Motion and Hyperbolic Geometry in Special Relativity

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1. Abstract

Relative motion is fundamental to the theory of relativity and for mechanical systems was used by Galileo (1632) and Huyghens (1656). Absolute velocity was defined by Newton (1687) who needed it for his treatment of acceleration and force. But Newton also made use of relative velocities and much of Newtonian mechanics can be stated in terms of them. It is recognized that the early development of relativity was guided by the idea of extending this mechanical relativity to cover electromagnetic phenomena, this being clearly seen in the case of Einstein. As a result of the historical development though, the common understanding is that relative motion in Special Relativity is similar to that of classical mechanics. However it is different and this fact tends to be not recognized because the relativity form can only be defined in a satisfactory way by using the hyperbolic (Lobachevskian) theory of relativity which is not described in current textbooks.

In the case of rectilinear motion, relative velocity is fairly easy to deal with by using the velocity composition rule applied to the difference of velocities but even here the essential properties are only clearly seen when formulae are expressed in terms of rapidity or hyperbolic velocity. Doing so results in better understanding of well known laws such as those for the Doppler effect, rectilinear percussion, Newton's law of motion.

In three dimensions, the general formula for relative velocity was first given by Fock (1955) who also showed its relation to hyperbolic geometry. In the present paper we describe a more general approach leading to Fock's definition and apply it to define acceleration, a calculation already implicit in the remarkable early paper by Thomas (1927).

2. Relative Velocities in Rectilinear Motion

Suppose two points P_1, P_2 are moving with velocities v_1, v_2 relative to an origin O and let u be the velocity of P_2, relative to P_1.

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O           P₁        P₂
o----------o---------o-------------
            v₁→       v₂→
…..u→
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Fig. Relative rectilinear motions

Transferring the origin from O to the moving point P_1 gives, by the composition rule

$$v_2 = \frac{v_1 + u}{1 + v_1.u/c^2} \tag{1}$$

Solving for u gives

$$u = \frac{v_2 - v_1}{1 + v_1.u / c^2} \tag{2}$$

for the relative velocity of two points moving with velocities v_2 and v_1 relative to origin O. It might appear that since v_2 and v_1 depend on the origin, then u must also but it is easily verified that this is not so.

Taking the formula (2) as a starting point it is possible to prove the formula for composition of relative velocities in a convenient way. Consider the situation in the figure where 3 points P_1, P_2, P_3 are in motion relative to an origin O.

O $\quad$ P_1 $\quad$ P_2 $\quad$ P_3

o----------o-------o-------o------

$\quad$ $v_1 \rightarrow$ $\quad$ $v_2 \rightarrow$ $\quad$ $v_3 \rightarrow$

Fig. Composition of relative rectilinear velocities

The relative velocities $u_{2/1}$ of P_2 to P_1 and $u_{3/2}$ of P_3 to P_2, are

$$u_{2/1} = \frac{v_2 - v_1}{1 - v_2.v_1/c^2} \qquad u_{3/2} = \frac{v_3 - v_2}{1 - v_3.v_2/c^2} \tag{3}$$

The velocity $u_{3/1}$ of P_3 relative to P_1 is found from

$$u_{3/1} = \frac{v_3 - v_1}{1 - v_3.v_1/c^2} \tag{4}$$

which algebraically is easily shown to be equivalent to

$$u_{3/1} = \frac{u_{3/2} + u_{2/1}}{1 + u_{3/2}.u_{2/1}/c^2} \tag{5}$$

Use of hyperbolic velocities: These relations become simplied and more transparent by the use of hyperbolic velocities (Varićak 1912). Hyperbolic velocity V is related to velocity v by the inverse hyperbolic relation

$$V = c\ \mathrm{th}^{[-1]} (v/c) \tag{6}$$

so that

$$v = c\ \mathrm{th}\ (V/c) \tag{7}$$

and is easy to verify that e.g. the above equations (1), (2) become in obvious notation

$$V_2 = V_1 + U \tag{8}$$

$$U = V_2 - V_1 \qquad (9)$$

With hyperbolic velocity it is easy to refer all velocities back to an origin. Suppose, as before, there are 3 moving points A, B, C referred to an origin O. Then

$$U_{2/1} = V_2 - V_1 \qquad (10)$$

$$U_{3/2} = V_3 - V_2 \qquad (11)$$

from which by addition,

$$U_{3/2} + U_{2/1} = V_3 - V_1 = U_{3/1} \qquad (12)$$

The use of hyperbolic velocity leads to a simplification of several laws in special relativity.

(a) It makes possible the reformulation of Einstein's Doppler shift law in a way completely analogous to the classical linear form by the use of a modified definition of redshift. (See the writer's 1992 PIRT paper suggested by work of Prokhovnik)
(b) It simplifies the laws of mechanical collisions bringing out the analogy with classical theory. (See next section)
(c) It leads to a formulation of Newton's 3rd Law of Motion in a way analogous to that of the classical form i.e. force is (rest) mass times acceleration. (See writer's PIRT papers of 2000 and 2002)

3. Translational Invariance – Galilean and Lorentzian.

Special relativity has to do with invariance of physical laws for inertial frames in uniform relative motion. The fact that uniform motion in the Galilean sense implies and is implied by uniform relative motion in the Lorentz sense blurs a distinction here because a Galilean velocity translation is not the same as a Lorentzian velocity translation. The use of hyperbolic velocity makes the distinction clearer. A Galilean translation transforms a velocity v by a shift u as

$$v \rightarrow v + u \qquad (1)$$

while a Lorentz translation transforms a hyperbolic velocity V by a shift U as

$$V \rightarrow V + U \qquad (2)$$

A simple illustration of the distinction is given by the collision of two masses. Relative to the line of approach the classical theory leaves invariant the quantities

$$\begin{aligned} m_1 + m_2 &= \text{const.} \\ m_1 v_1 + m_2 v_2 &= \text{const.} \end{aligned} \qquad (3)$$

Or equivalently we can say that

$$m_1 (v_1 - v) + m_2 (v_2 - v) = m_1 (v'_1 - v) + m_2 (v'_2 - v) \quad (4)$$

where the dash denotes the value after the collision and v is an arbitrary velocity e.g. the velocity of the centre of gravity.

In special relativity the situation is analogous though not the same. The governing equations are (Lewis & Tolman 1908 etc) the *conservation of mass-energy*:

$$\frac{m_1}{\sqrt{(1-v_1^2/c^2)}} + \frac{m_2}{\sqrt{(1-v_2^2/c^2)}} = \frac{m_1}{\sqrt{(1-v'^2_1/c^2)}} + \frac{m_2}{\sqrt{(1-v'^2_2/c^2)}}. \quad (5)$$

and the *conservation of momentum*:

$$\frac{m_1 v_1}{\sqrt{(1-v_1^2/c^2)}} + \frac{m_2 v_2}{\sqrt{(1-v_2^2/c^2)}} = \frac{m_1 v'_1}{\sqrt{(1-v'^2_1/c^2)}} + \frac{m_2 v'_2}{\sqrt{(1-v'^2_2/c^2)}} \quad (6)$$

On rewriting the equations in terms of hyperbolic velocities these become

$$m_1 \operatorname{ch} V_1/c + m_2 \operatorname{ch} V_2/c = m_1 \operatorname{ch} V'_1/c + m_2 \operatorname{ch} V'_2/c$$
$$m_1 \operatorname{sh} V_1/c + m_2 \operatorname{sh} V_2/c = m_1 \operatorname{sh} V'_1/c + m_2 \operatorname{sh} V'_2/c \quad (7)$$

These equations will hold in any uniformly moving frame of reference if:

$$m_1 \operatorname{ch} (V_1 - V)/c + m_2 \operatorname{ch} (V_2 - V)/c = m_1 \operatorname{ch} (V'_1 - V)/c + m_2 \operatorname{ch} (V'_2 - V)/c$$
$$m_1 \operatorname{sh} (V_1 - V)/c + m_2 \operatorname{sh} (V_2 - V)/c = m_1 \operatorname{sh} (V'_1 - V)/c + m_2 \operatorname{sh} (V'_2 - V)/c$$

(8)

The quantities written in the last two equations represent the mass-energy and momentum relative to a frame having hyperbolic velocity V. On expanding the hyperbolic functions we see that these equations will always be satisfied if the initial two equations for mass-energy and momentum are satisfied. Conversely, if this principle of relativity holds then either of the equations implies the other. For a different view of this example see the papers of Compte PIRT 1998, 2004.

4. Fock's Definition of Relative Velocity

Relative velocity for three dimensional motion was defined by Fock (1955). Suppose there are two points P_1, P_2 moving with velocities $\mathbf{v}_1$, $\mathbf{v}_2$ relative to an origin O. Fock defined the velocity of P_2 relative to P_1 by transferring the origin from O to P_1 so reducing P_1 to rest. The velocity of P_2 with respect to this new origin is then defined as the velocity $\mathbf{v}_{2/1}$ of P_2 relative to P_1.

The move to the new origin at P_1 corresponds to transformation to coordinates $\mathbf{r}'$, t ' with differential relations written in vector notation

$$dt' = \gamma_1 (dt - d\mathbf{r}.\mathbf{v} /c^2)$$

$$d\mathbf{r}' = d\mathbf{r} - \mathbf{v}_1\, dt + (\gamma_1 - 1)\, \mathbf{n}_1(\mathbf{n}_1 . d\mathbf{r} - v_1\, dt) \qquad (1)$$

where $\mathbf{n}_1$ is a unit vector in the direction of $\mathbf{v}_1$ and factor γ_1 refers to this velocity:

$$\gamma_1 = 1/\{1 - (v_1/ c)^2\} \qquad (2)$$

Then by division the relative velocity $\mathbf{v}_{2/1}$ is found as the value

$$\frac{d\mathbf{r}'}{dt'} = \frac{\mathbf{v}_2 - \mathbf{v}_1 + (\gamma_1 - 1)\, \mathbf{n}_1.\{(\mathbf{n}_1.\mathbf{v}_2) - v_1\}}{\gamma_1(1 - \mathbf{v}_1.\mathbf{v}_2 / c^2)} \qquad (3)$$

When $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same direction, $\mathbf{n}_1\,(\mathbf{n}_1.\mathbf{v}_2)$ becomes $\mathbf{v}_2$ and the formula reduces to the rectilinear case:

There will be a similar expression for the velocity $\mathbf{v}_{1/2}$ of P_1 relative to P_2 but the two relative velocities are not the negatives of each other. They do however have the same magnitude given by Einstein's combination formula for the difference of the velocities.

Derivation: Making use of the convenient dyadic notation, the numerator is

$$\mathbf{v}_2 - \mathbf{v}_1 + (\gamma_1 - 1)\, \mathbf{n}_1.\{(\mathbf{n}_1.\mathbf{v}_2) - v_1\} = \mathbf{v}_2 - \mathbf{v}_1 + (\gamma_1 - 1)\, \mathbf{n}_1\mathbf{:}\mathbf{n}_1\, (\mathbf{v}_2 - \mathbf{v}_1)$$

$$= \gamma_1\, [\mathbf{n}_1\mathbf{:}\mathbf{n}_1(\mathbf{v}_2 - \mathbf{v}_1) + \sqrt{(1 - (v_1/c)^2)}\, (1 - \mathbf{n}_1\mathbf{:}\mathbf{n}_1)(\mathbf{v}_2 - \mathbf{v}_1)\,] \qquad (4)$$

where in the square bracket, the classical expression for relative velocity $\mathbf{v}_2$-$\mathbf{v}_1$ is resolved into components in the direction of and transverse to $\mathbf{v}_1$ with a contraction of the transverse component. From this it follows that the magnitude squared is

$$(v_2 \cos\theta - v_1)^2 + (1 - (v_1/c)^2)\, v_2^2 \sin^2\theta$$
$$= v_2^2 - 2v_1v_2 \cos\theta + v_1^2 - (v_1\, v_2/c)^2 \sin^2\theta \qquad (5)$$

Restoring the denominator gives Einstein's composition rule for magnitude squared of the difference of velocities.

$$\frac{v_2^2 - 2v_1v_2 \cos\theta + v_1^2 - (v_1\, v_2/c)^2 \sin^2\theta}{(1 - v_1\, v_2/ c^2)^2} \qquad (6)$$

This can also be expressed in vector form

$$\frac{|\, \mathbf{v}_2 - \mathbf{v}_1\, |^2 - |\, \mathbf{v}_1 \mathrm{x}\, \mathbf{v}_2\, |^2 / c^2}{(1 - \mathbf{v}_1\, \mathbf{v}_2 / c^2)^2} \qquad (7)$$

The result is symmetrical in the two velocities showing that $\mathbf{v}_{1/2}$ and $\mathbf{v}_{2/1}$ have the same magnitude.

5. Alternative Method of Definition of Relative Velocity.

A more general approach to the definition of relative velocity based on the same idea as was used earlier in this article for one dimensional motion is the following.

First we take as the standard form for the Lorentz translation for velocity **v** = (v_1, v_2, v_3)

$$\begin{bmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v_1/c & -\gamma v_2/c & -\gamma v_3/c \\ -\gamma v_1/c & 1+(\gamma-1)n_1^{\ 2} & (\gamma-1)n_1 n_2 & (\gamma-1)n_1 n_3 \\ -\gamma v_2/c & (\gamma-1)n_2 n_1 & 1+(\gamma-1)n_2^{\ 2} & (\gamma-1)n_2 n_3 \\ -\gamma v_3/c & (\gamma-1)n_3 n_1 & (\gamma-1)n_3 n_2 & 1+(\gamma-1)n_3^{\ 2} \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} \tag{1}$$

where **n** = (n_1, n_2, n_3) = **v**/v is a unit vector in the direction of the velocity. This relation can be written more concisely using partitioned matrices as

$$\begin{bmatrix} cdt' \\ \mathbf{dr'} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma \mathbf{v}^T/c \\ -\gamma \mathbf{v}/c & I+(\gamma-1)\mathbf{nn}^T/c^2 \end{bmatrix} \begin{bmatrix} cdt \\ \mathbf{dr} \end{bmatrix} \tag{2}$$

where bold letters are used for 3x1 column vectors for emphasis. The Lorentz matrix here will be denoted by Λ (**v**)

<u>The definition of relative velocity</u> is then as follows. It is assumed again that there is an observer O and the two moving points P_1.P_2 whose relative velocity we wish to define. The use of suffices will be changed to refer to these points. Then the relations between coordinate changes are, when referred to the observer O,

$$\begin{bmatrix} cdt_1 \\ \mathbf{dr}_1 \end{bmatrix} = [\Lambda(\mathbf{v}_1)] \begin{bmatrix} cdt_0 \\ \mathbf{dr}_0 \end{bmatrix}, \qquad \begin{bmatrix} cdt_2 \\ \mathbf{dr}_2 \end{bmatrix} = [\Lambda(\mathbf{v}_2)] \begin{bmatrix} cdt_0 \\ \mathbf{dr}_0 \end{bmatrix} \tag{3}$$

from which is deduced

$$\begin{bmatrix} cdt_2 \\ \mathbf{dr}_2 \end{bmatrix} = [\Lambda(\mathbf{v}_2)][\Lambda(\mathbf{v}_1)]^{-1} \begin{bmatrix} cdt_1 \\ \mathbf{dr}_1 \end{bmatrix} \tag{4}$$

Although the matrices Λ($\mathbf{v}_2$), Λ ($\mathbf{v}_1$) are defined relative to the observer O, the product of the matrices here is independent of O. For suppose that the relation between the coordinate changes between the two observers O, O' is

$$\begin{bmatrix} cdt_0 \\ \mathbf{dr}_0 \end{bmatrix} = [\Lambda(\mathbf{v}_0)] \begin{bmatrix} cdt'_0 \\ \mathbf{dr}'_0 \end{bmatrix} \tag{5}$$

then the matrices $\Lambda(\mathbf{v}_2)$, $\Lambda(\mathbf{v}_1)$ will transform by

$$\Lambda(\mathbf{v}_2) \rightarrow \Lambda(\mathbf{v}_2)\,\Lambda(\mathbf{v}_0), \quad \Lambda(\mathbf{v}_1) \rightarrow \Lambda(\mathbf{v}_1)\,\Lambda(\mathbf{v}_0) \qquad (6)$$

leaving the product $\Lambda(\mathbf{v}_2)\,\Lambda(\mathbf{v}_1)^{-1}$ unchanged.

Composition of relative velocities: Let there be three points P_1, P_2, P_3 moving with velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ relative to the observer O. The relations

$$\begin{aligned}\Lambda(\mathbf{v}_{2/1}) &= \Lambda(\mathbf{v}_2)\,\Lambda(-\mathbf{v}_1)\\ \Lambda(\mathbf{v}_{3/2}) &= \Lambda(\mathbf{v}_3)\,\Lambda(-\mathbf{v}_2)\\ \Lambda(\mathbf{v}_{3/1}) &= \Lambda(\mathbf{v}_3)\,\Lambda(-\mathbf{v}_1)\end{aligned} \qquad (7)$$

from which follows the basic composition rule:

$$\Lambda(\mathbf{v}_{3/1}) = \Lambda(\mathbf{v}_{3/2})\,\Lambda(\mathbf{v}_{2/1}) \qquad (8)$$

Deduction of Fock's expression for relative velocity: The product $\Lambda(\mathbf{v}_2)\,\Lambda(\mathbf{v}_1)^{-1}$ may also be written as $\Lambda(\mathbf{v}_2)\Lambda(-\mathbf{v}_1)$ and evaluated as a product of Lorentz transformations which can be written in the form R $\Lambda(\mathbf{v})$ where R is a spatial rotation and $\Lambda(\mathbf{v})$ a Lorentz translation. More explicitly it is

$$\begin{bmatrix}1 & 0\\ 0 & \Omega\end{bmatrix}\begin{bmatrix}\gamma & -\gamma\mathbf{v}^T/c\\ -\gamma\mathbf{v}/c & I+(\gamma-1)\mathbf{n}\mathbf{n}^T\end{bmatrix} \qquad (9)$$

where Ω is a 3x3 spatial rotation matrix. Then the equation for the relative velocity matrix $\Lambda(\mathbf{v})$ and rotation matrix R is:

$$R\,\Lambda(\mathbf{v}) = \Lambda(\mathbf{v}_2)\,\Lambda(-\mathbf{v}_1) \qquad (10)$$

Performing the calculation the parameters of the product are found as

$$\gamma = \gamma_1\gamma_2\{1 - \mathbf{v}_1^T\mathbf{v}_2/c^2\}$$

$$\gamma.\mathbf{v} = \gamma_2\{(I + (\gamma_1-1)\,\mathbf{n}_1\mathbf{n}_1^T)\,\mathbf{v}_2 - \gamma_1\,\mathbf{v}_1\}$$

$$\gamma.\Omega\,\mathbf{v} = \gamma_1\{(I + (\gamma_2-1)\,\mathbf{n}_2\mathbf{n}_2^T)(-\mathbf{v}_1) + \gamma_2\,\mathbf{v}_2\}$$

$$\Omega + (\gamma - 1)(\Omega.\mathbf{n})\mathbf{n} = (I + (\gamma_2-1)\,\mathbf{n}_2\mathbf{n}_2^T)(I + (\gamma_1-1)\,\mathbf{n}_1\mathbf{n}_1^T) - \gamma_2\gamma_1\,\mathbf{v}_2\mathbf{v}_1^T/c^2$$

$$(11)$$

From the first and second of these equations we find

$$\mathbf{v} = \frac{\{I + (\gamma_1-1)\,\mathbf{n}_1\mathbf{n}_1^T\}\,\mathbf{v}_2 - \gamma_1\,\mathbf{v}_1}{\gamma_1\{1 - \mathbf{v}_1^T\mathbf{v}_2/c^2\}} \qquad (12)$$

which represents the relative velocity $v_{2/1}$ of P_2 relative to P_1. So that, rearranging slightly

$$\mathbf{v}_{2/1} = \frac{\mathbf{v}_2 - \mathbf{v}_1 + (\gamma_1 - 1)\,\mathbf{n}_1\{(\mathbf{n}_1^T\,\mathbf{v}_2) - v_1\}}{\gamma_1(1 - \mathbf{v}_1^T\,\mathbf{v}_2/c^2)} \qquad (13)$$

Interchange of the suffixes gives the reverse relative velocity:

$$\mathbf{v}_{1/2} = \frac{\mathbf{v}_1 - \mathbf{v}_2 + (\gamma_2 - 1)\,\mathbf{n}_2\{(\mathbf{n}_2^T\mathbf{v}_1) - v_2\}}{\gamma_2(1 - \mathbf{v}_2^T\mathbf{v}_1/c^2)} \qquad (14)$$

There is the associated spatial rotation R which Fock did not mention which is the reason for the fact that the two relative velocities are not negatives of one another.

6. Relative Velocity and Hyperbolic Geometry

To Fock belongs the credit of having realized that the correct representation of the all-important concept of relative velocity in Special Relativity requires the use of hyperbolic space. The appropriate representation of this space is that of Beltrami (see Beltrami's very readable 1868 article reproduced in translation by Stillwell 1996).

Now as seen above, the magnitude of the relative velocity between two velocities $\mathbf{v}_1$, $\mathbf{v}_2$ is given by the Einstein composition rule for the difference of these vectors and by rearrangment of this formula there is immediately deduced that the magnitudes v_1, v_2 are related to the magnitude v of a resultant by

$$1 - \frac{v^2}{c^2} = \frac{(1 - v_1^2/c^2)(1 - v_2^2/c^2)}{(1 - \mathbf{v}_1.\mathbf{v}_{2/}/c^2)} \qquad (1)$$

Inverting this relation we find

$$\frac{1}{1 - v^2/c^2} = \frac{c^2 - \mathbf{v}_1\,\mathbf{v}_2}{(c^2 - v_1^2)(c^2 - v_2^2)}. \qquad (2)$$

which, on using the angle θ between $\mathbf{v}_1$ and $\mathbf{v}_2$ and the hyperbolic velocities V, V_1, V_2 corresponding to $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$ gives

$$\mathrm{ch}\, V/c = \frac{c^2 - v_1 v_2 \cos\theta}{(c^2 - v_1^2)(c^2 - v_2^2)\ \sqrt{}\,(c^2 - v_1^2)(c^2 - v_2^2)}.$$

$$= \mathrm{ch}\, V_1/c\ \mathrm{ch}\, V_2/c - \mathrm{sh}\, V_1/c\ \mathrm{sh}\, V_2/c \cos\theta \qquad (3)$$

This equation is what in hyperbolic trigonometry corresponds to the cosine rule in spherical trigonometry. The relationships between V, V_1, V_2 may consequently be represented by the sides of a triangle in hyperbolic space.

The Beltrami representation: possible velocities (v_x, v_y, v_z) in Special Relativity are restricted by the inequality

$$v^2 = v_x^2+v_y^2+v_z^2 < c^2 \qquad (4)$$

and the velocity space is consequently represented by vectors drawn from the origin to points inside a sphere of radius c.

The equation for V/c given above defines the Beltrami metric giving distance between two velocities $\mathbf{v}_1$, $\mathbf{v}_2$ in this equation. With this metric, the interior of the space of admissible velocities becomes a hyperbolic space

7. Acceleration and Thomas rotation

Suppose a moving point has, relative to some observer O, a velocity $\mathbf{v}$ at time t and $\mathbf{v} + \delta\mathbf{v}$ at time t + δt. To define acceleration we need to know the increment in velocity relative to the moving point, i.e. the relative velocity for the times t, t + δ t. Applying the previous definition giving relative velocity of the increment we have to consider the value of the product

$$\Lambda(\mathbf{v} + \delta\mathbf{v})\ \Lambda(-\mathbf{v}) \qquad (1)$$

The answer was given in the second paper of Thomas (1927) which contains, with a slight notational difference, the following statement:

*"A Lorentz transformation with velocity - **v** followed by one with velocity **v+δv**, where **δv** is infinitesimal, resolves into a Lorentz transformation with infinitesimal velocity*

$$\gamma\{\boldsymbol{\delta v} + (\gamma\text{-}1)\ (\boldsymbol{v.\delta v})\ \boldsymbol{v} / v^2\}$$

together with an infinitesimal rotation

$$(\gamma\text{-}1)\ (\boldsymbol{v x \delta v}) / v^2 \quad " \qquad (2)$$

The first expression here is the relative velocity in the sense of Fock for the special case of infinitesimal velocity increment. It is remarkable that Thomas already in 1927 had come to the formula for relative velocity for this special case.

To verify this assertion, replace in Fock's formula, $\mathbf{v}_1$, $\mathbf{v}_2$ by $\mathbf{v}$ and $\mathbf{v+\delta v}$ when it becomes to first order,

$$\frac{\boldsymbol{\delta}\mathbf{v} + (\gamma - 1)\ \mathbf{n}\ (\mathbf{n}.\boldsymbol{\delta}\mathbf{v})}{\gamma\ (1 - v^2/c^2)} = \gamma\ \{\boldsymbol{\delta}\mathbf{v} + (\gamma - 1)\ \mathbf{n}\ (\mathbf{n}.\boldsymbol{\delta}\mathbf{v})\} \qquad (3)$$

as stated by Thomas. Here $\mathbf{n}$ is the unit vector for $\mathbf{v}$. This expression can be rewritten as

$$\gamma^2\ \mathbf{n{:}n}.\boldsymbol{\delta}\mathbf{v} + \gamma\ (1\text{-}\ \mathbf{n{:}n})\boldsymbol{\delta}\mathbf{v} = \gamma^2\ \mathbf{n}\ \delta v_1 + \gamma\ \mathbf{n}^{\perp}\ \delta v_2 \qquad (4)$$

$\mathbf{n}^{\perp}$ being perpendicular to **n** and δv_1, δv_2 are the components of **v** in directions **n**, $\mathbf{n}^{\perp}$. The acceleration **α** is now found by dividing by dτ or equally by dt $\sqrt{(1 - v^2/c^2)}$ giving.

$$\boldsymbol{\alpha} = \gamma^3 \mathbf{n} \frac{dv_1}{dt} + \gamma^2 \mathbf{n}^{\perp} \frac{dv_2}{dt} = \gamma^3 \{ \mathbf{n} \frac{dv_1}{dt} + \sqrt{(1 - v^2/c^2)}\, \mathbf{n}^{\perp} \frac{dv_2}{dt} \} \qquad (5)$$

We see here the characteristic transverse contraction. As stated by Thomas, there is in addition a rotation angle

$$(\gamma\text{-}1)(\mathbf{v}\mathbf{x}\boldsymbol{\delta}\mathbf{v})/v^2 \qquad (6)$$

giving the Thomas rotation with vectorial angular velocity

$$(\gamma\text{-}1)(\mathbf{v}\ \mathbf{x}\ d\mathbf{v}/dt)/v^2 \qquad (7)$$

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